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CONTINUOUS SOLUTIONS OF ITERATIVE EQUATIONS OF INFINITE ORDER

Abstract. Given a probability space $(\Omega, \mathcal{A}, P)$ and a complete separable metric space X, we consider continuous and bounded solutions $\varphi\colon X \to \mathbb{R}$ of the equations $\varphi(x) = \int_\Omega \varphi(f(x,\omega))P(d\omega)$ and $\varphi(x) = 1 - \int_\Omega \varphi\big(f(x,\omega)\big)P(d\omega)$, assuming that the given function $f\colon X \times \Omega \to X$ is controlled by a random variable $L\colon \Omega \to (0,\infty)$ with $-\infty < \int_\Omega \log L(\omega)P(d\omega) < 0$. An application to a refinement type equation is also presented.

Keywords: random-valued vector functions, sequences of iterates, iterative equations, continuous solutions.

Mathematics Subject Classification: Primary 45A05, 39B12; Secondary 39B52, 60B12.

1. INTRODUCTION

Throughout this paper we assume that $(\Omega, \mathcal{A}, P)$ is a probability space, (X, d) is a complete separable metric space and $f\colon X \times \Omega \to X$ is a random-valued function, i.e., it is measurable with respect to the product σ-algebra $\mathcal{B}(X) \otimes \mathcal{A}$, where $\mathcal{B}(X)$ denotes the σ-algebra of all Borel subsets of X. We consider the equation

$$\varphi(x) = \int_\Omega \varphi\big(f(x,\omega)\big)P(d\omega), \tag{1.1}$$

which has extensively been studied in various classes of functions (see, e.g., [3, 7, 13]). For more details concerning equation (1.1) and its particular cases, we refer the reader to survey papers [2, part 4] and [1]. Following [11], we also examine the equation of the form

$$\varphi(x) = 1 - \int_\Omega \varphi\big(f(x,\omega)\big)P(d\omega). \tag{1.2}$$

Numerous papers concern equation (1.1) with $f(x,\omega) = L(\omega)x - M(\omega)$, assuming that $0 < \int_\Omega \log L(\omega)P(d\omega) < \infty$. In the present paper we are interested in the opposite case

$$-\infty < \int\limits_\Omega \log L(\omega)P(d\omega) < 0. \tag{1.3}$$

More precisely, we adopt the following hypothesis.

(H) There is a measurable function $L\colon \Omega \to (0,\infty)$ such that

$$d(f(x,\omega), f(y,\omega)) \le L(\omega)d(x,y) \quad \text{for } x,y \in X, \omega \in \Omega \tag{1.4}$$

and (1.3) holds.

As an application of the results obtained, we get a corollary on L^1-solutions of the equation

$$\Phi(x) = \int\limits_\Omega |\det A(\omega)F'(x)|\Phi(A(\omega)F(x) - C(\omega))P(d\omega). \tag{1.5}$$

Equation (1.5) extends both the discrete and the continuous refinement equations which have extensively been studied in connection with their applications (see, e.g., [5, 6, 8, 16]).

The presented results are related to invariance properties of the transfer operator for Markov chains associated with iterated random functions (see, e.g., [9]). In fact, the probability distribution of the limit of the sequences of iterates of a random function satisfies (1.1). Our purpose is to investigate solutions of (1.1), as well as (1.2), in wider classes of functions; e.g., in the class of bounded and continuous functions.

2. MAIN RESULTS

We begin with the following simple lemma.

Lemma 2.1. *If* (1.3) *holds, then the sequence* $\left(\prod_{n=1}^N L(\omega_n)\right)$ *converges a.s. to zero.*

Proof. By the Kolmogorov strong law of large numbers,

$$\lim_{N\to\infty} \Big(\prod_{n=1}^N L(\omega_n)\Big)^{\frac{1}{N}} = \exp\left\{\int\limits_\Omega \log L(\omega)P(d\omega)\right\} < 1 \quad \text{a.s.}$$

Consequently,

$$\lim_{N\to\infty} \prod_{n=1}^N L(\omega_n) = 0 \quad \text{a.s.}$$

□

In the proofs of our results, we will iterate the random-valued function f. The iterates of such a function are defined by (see [4, 10])

$$f^1(x,\omega_1,\omega_2,\dots)=f(x,\omega_1),\quad f^{n+1}(x,\omega_1,\omega_2,\dots)=f(f^n(x,\omega_1,\omega_2,\dots),\omega_{n+1}).$$

Note that f^n is a random-valued function on the product probability space $(\Omega^\infty,\mathcal{A}^\infty,P^\infty)$.

We are now in a position to formulate our results. First note that the unique constant solution of (1.2) equals 1/2 and we will omit this simple fact in all results of this section.

Proposition 2.2. *Assume* **(H)** *and let* (σ_n) *be a sequence of measure preserving transformations of* $(\Omega^\infty,\mathcal{A}^\infty,P^\infty)$ *such that*

$$\bigwedge_{\omega\in\Omega^\infty}\Big[\Big(\bigwedge_{m\in\mathbb{N}}\lim_{N\to\infty}\prod_{n=1}^{N}L((\sigma_m(\omega))_n)=0\Big)\Rightarrow\lim_{N\to\infty}\prod_{n=1}^{N}L((\sigma_N(\omega))_n)=0\Big].\tag{2.1}$$

If $x_0\in X$ *and if* $(f^n(x_0,\cdot)\circ\sigma_n)$ *has a subsequence which converges in measure, then every continuous and bounded solution* $\varphi\colon X\to\mathbb{R}$ *of* (1.1) *or* (1.2) *is constant.*

Proof. Put

$$A=\bigcap_{m=1}^{\infty}\sigma_m^{-1}\left(\left\{\omega\in\Omega^\infty\colon\lim_{N\to\infty}\prod_{n=1}^{N}L(\omega_n)=0\right\}\right).$$

From Lemma 2.1 it follows that $P^\infty(A)=1$. By (2.1),

$$\lim_{N\to\infty}\prod_{n=1}^{N}L((\sigma_N(\omega))_n)=0\qquad\text{for }\omega\in A.$$

Using (1.4) and a simple induction, we obtain

$$d\big(f^N(x,\sigma_N(\omega)),f^N(y,\sigma_N(\omega))\big)\le d(x,y)\prod_{n=1}^{N}L((\sigma_N(\omega))_n)\tag{2.2}$$

for $x,y\in X,\omega\in\Omega^\infty,N\in\mathbb{N}$.

Assume now that $(f^{n_k}(x_0,\cdot)\circ\sigma_{n_k})$ converges in measure. Without loss of generality, we can assume that (n_k) contains even (or odd) numbers only. From (2.2) it follows that for every $x\in X$ the sequence $(f^{n_k}(x,\cdot)\circ\sigma_{n_k})$ converges in measure and the limit ξ is independent of x.

Let $\varphi\colon X\to\mathbb{R}$ be a continuous and bounded solution of (1.1) or (1.2). In both cases

$$\varphi(x)=\int_{\Omega^\infty}\varphi(f^{2n}(x,\omega))P^\infty(d\omega),$$

whence

$$\varphi(x) = \int_{\Omega^\infty} \varphi(f^{2n}(x, \sigma_{2n}(\omega)))P^\infty(d\omega)$$

for $x \in X, n \in \mathbb{N}$. Passing to the limit, we get

$$\varphi(x) = \int_{\Omega^\infty} \varphi(\xi(\omega))P^\infty(d\omega) \qquad \text{for } x \in X,$$

which shows that φ is constant. □

The following result gives some condition on f under which the sequence $(f^n(x, \cdot) \circ \sigma_n)$ converges a.s. for a special sequence (σ_n).

Theorem 2.3. *Assume* **(H)** *and let* $x_0 \in X$. *If*

$$\int_\Omega \log \max\{d(f(x_0, \omega), x_0), 1\}P(d\omega) < \infty, \tag{2.3}$$

then every continuous and bounded solution $\varphi\colon X \to \mathbb{R}$ *of* (1.1) *or* (1.2) *is constant.*

Proof. Following [14], define a sequence (σ_n) by

$$\sigma_n(\omega_1, \omega_2, \dots) = (\omega_n, \dots, \omega_1, \omega_{n+1}, \dots).$$

Clearly, σ_n preserves the product measure P^∞ and (2.1) holds. According to Proposition 2.2, it is enough to show the convergence of $(f^n(x_0, \cdot) \circ \sigma_n)$. Since $f^n(\cdot, \omega)$ depends exclusively on the first n coordinates of $\omega \in \Omega^\infty$, we see that (2.2) implies

$$d\big(f^{N+1}(x_0, \sigma_{N+1}(\omega)), f^N(x_0, \sigma_N(\omega))\big) \le \prod_{n=1}^{N} L(\omega_n)d(f(x_0, \omega_{N+1}), x_0),$$

whence

$$d(f^{N+N'}(x_0, \sigma_{N+N'}(\omega)), f^N(x_0, \sigma_N(\omega))) \le \sum_{n=N}^{N+N'-1} \prod_{k=1}^{n} L(\omega_k)d(f(x_0, \omega_{n+1}), x_0)$$

for $\omega \in \Omega^\infty, N, N' \in \mathbb{N}$. Consequently, in view of [11, Theorem 2] and (2.3), the series

$$\sum_{N=1}^{\infty} \prod_{n=1}^{N} L(\omega_n)d(f(x_0, \omega_{N+1}), x_0)$$

converges almost surely on Ω^∞ and the required convergence follows. □

Theorem 2.4. *If* **(H)** *holds, then every bounded and uniformly continuous function* $\varphi\colon X \to \mathbb{R}$ *satisfying*

$$|\varphi(x) - \varphi(y)| \le \int_\Omega |\varphi(f(x,\omega)) - \varphi(f(y,\omega))| P(d\omega) \qquad \text{for } x, y \in X \tag{2.4}$$

is constant.

Proof. Let $\varphi\colon X \to (-M, M)$ be a uniformly continuous function such that (2.4) holds.

Fix $x, y \in X, \varepsilon > 0$ and let δ be a positive real such that $|\varphi(u) - \varphi(v)| \le \frac{\varepsilon}{2}$, provided $d(u,v) \le \delta$ for $u, v \in X$.

From (1.4) and Lemma 2.1, we infer $\lim_{N\to\infty} d(f^N(x,\omega), f^N(y,\omega)) = 0$. Hence, for a sufficiently large $N \in \mathbb{N}$ and for suitably chosen set $A \in \mathcal{A}^\infty$, there holds

$$P^\infty(\Omega^\infty \setminus A) \le \frac{\varepsilon}{4M} \qquad \text{and} \qquad d(f^N(x,\omega), f^N(y,\omega)) \le \delta \qquad \text{for } \omega \in A.$$

Finally, by iterating (2.4), we obtain

$$|\varphi(x) - \varphi(y)| \le \int_A |\varphi(f^N(x,\omega)) - \varphi(f^N(y,\omega))| P^\infty(d\omega) + 2MP^\infty(\Omega^\infty \setminus A) \le \varepsilon,$$

which completes the proof. □

As a consequence of Theorem 2.4, we obtain a result concerning the uniqueness in the class of uniformly continuous and bounded functions.

Corollary 2.5. *If* **(H)** *holds, then every bounded and uniformly continuous solution* $\varphi\colon X \to \mathbb{R}$ *of* (1.1) *or* (1.2) *is constant.*

The next two examples show that neither boundedness nor continuity may be omitted in Theorems 2.3, 2.4 and in Corollary 2.5.

Example 2.6. If

$$f(0,\omega) = 0 \qquad \text{and} \qquad f(x,\omega) \ne 0 \qquad \text{for } x \ne 0, \omega \in \Omega,$$

then (2.3) holds with $x_0 = 0$ and for every $\alpha, \beta \in \mathbb{R}$ with $\alpha \ne \beta$, the function

$$\varphi = \alpha\chi_{\{0\}} + \beta\chi_{\mathbb{R}\setminus\{0\}} \tag{2.5}$$

is a bounded and discontinuous solution of (1.1). If $\alpha + \beta = 1$, then (2.5) is a solution of (1.2), provided

$$f(0,\omega) \ne 0 \qquad \text{and} \qquad f(x,\omega) = 0 \qquad \text{for } x \ne 0, \omega \in \Omega.$$

Example 2.7. Let $\Omega = \{\omega_1, \omega_2\}$, let p_1, p_2 be positive reals with $p_1 + p_2 = 1$ and let $L_1 > 0$ satisfy

$$L_1^{p_1}\big(1 - p_1 L_1\big)^{p_2} < p_2^{p_2} \qquad \text{and} \qquad p_1 L_1 < 1.$$

Put

$$L_2 = \frac{1 - p_1 L_1}{p_2}, \qquad L(\omega_i) = L_i \qquad \text{and} \qquad f(x, \omega_i) = L(\omega_i)x$$

for $x \in \mathbb{R}, i = 1, 2$. Clearly, conditions (1.4) and (1.3) are fulfilled. Equation (1.1) now takes the form

$$\varphi(x) = p_1\varphi(L_1 x) + p_2\varphi(L_2 x).$$

Since $p_1L_1 + p_2L_2 = 1$, the identity function is a solution of the equation above. It is easy to verify that the function "$x \mapsto x + 1/2$" satisfies

$$\varphi(x) = 1 - p_1\varphi(-L_1 x) - p_2\varphi(-L_2 x).$$

Denote by $\mathbb{R}^{n\times m}$ the set of all matrices with n rows and m columns, and by $\|\cdot\|$ the maximum norm in $\mathbb{R}^n$.

From now on we assume that

$$f(x, \omega) = A(\omega)F(x) - C(\omega), \tag{2.6}$$

where $A = [A_{ij}]\colon \Omega \to \mathbb{R}^{n\times m}$, $C\colon \Omega \to \mathbb{R}^n$ are measurable and $F = [F_i]\colon \mathbb{R}^n \to \mathbb{R}^m$ is continuous. It is clear that the function given by (2.6) is random-valued (see [12]). Equations (1.1) and (1.2) now take the forms

$$\varphi(x) = \int_\Omega \varphi(A(\omega)F(x) - C(\omega))P(d\omega) \tag{2.7}$$

and

$$\varphi(x) = 1 - \int_\Omega \varphi(A(\omega)F(x) - C(\omega))P(d\omega), \tag{2.8}$$

respectively.

The following corollary will be useful in the next section.

Corollary 2.8. *Let $F(0) = 0$,*

$$|F_i(x) - F_i(y)| \le \|x - y\| \qquad \text{for } x, y \in X, i = 1, \dots, m$$

and

$$-\infty < \int_\Omega \log \max_{k=1,\dots,n} \{|A_{k1}(\omega)| + \dots + |A_{km}(\omega)|\} P(d\omega) < 0.$$

Then:

(i) *Every bounded and uniformly continuous solution $\varphi\colon \mathbb{R}^n \to \mathbb{R}$ of (2.7) or (2.8) is constant.*

(ii) *If*

$$\int_\Omega \log \max\{\|C(\omega)\|, 1\} P(d\omega) < \infty, \tag{2.9}$$

then every continuous and bounded solution $\varphi\colon \mathbb{R}^n \to \mathbb{R}$ of (2.7) or (2.8) is constant.

Proof. Clearly, (**H**) holds with

$$L(\omega) = \max_{k=1,\dots,n} \{|A_{k1}(\omega)| + \cdots + |A_{km}(\omega)|\}$$

and by (2.9) we obtain (2.3) with $x_0 = 0$. Hence the assertions follow from Corollary 2.5 and Theorem 2.3, respectively. □

3. AN APPLICATION TO A REFINEMENT TYPE EQUATION

Let $A\colon \Omega \to \mathbb{R}^{n\times n}$ and $C\colon \Omega \to \mathbb{R}^n$ be measurable, $\det A(\omega) \neq 0$ for $\omega \in \Omega$ and let $F\colon \mathbb{R}^n \to \mathbb{R}^n$ be a diffeomorphism. Then for an f of form (2.6), there holds

$$l_n \otimes P(f^{-1}(B)) = \int_\Omega l_n\Big(F^{-1}\big(A(\omega)^{-1}(B + C(\omega))\big)\Big)P(d\omega) = 0$$

for $B \in \mathcal{B}(\mathbb{R}^n)$ of zero Lebesgue measure l_n. Consequently, if $\Phi\colon \mathbb{R}^n \to \mathbb{R}$ is Lebesgue measurable, then $\Phi \circ f$ is measurable with respect to the completion of the product σ-algebra $\mathcal{L}_n \otimes \mathcal{A}$. Moreover, if the measure P is complete, then equation (1.5) with unknown L^1-function $\Phi\colon \mathbb{R}^n \to \mathbb{R}$ makes sense. (We omit details, which may be found in [15] for $n = 1$).

Fix measurable functions $a_1, \dots, a_n, c_1, \dots, c_n\colon \Omega \to \mathbb{R}$ and diffeomorphisms $F_1, \dots, F_n$ from $\mathbb{R}$ onto itself such that

$$F_i(0) = 0 \quad \text{and} \quad |F_i(x) - F_i(y)| \le |x - y| \quad \text{for } x, y \in \mathbb{R}, i = 1, \dots, n,$$

and define functions $A = [A_{ij}]\colon \Omega \to \mathbb{R}^{n\times n}$, $F\colon \mathbb{R}^n \to \mathbb{R}^n$ and $C\colon \Omega \to \mathbb{R}^n$ putting

$$F(x) = (F_1(x_1), \dots, F_n(x_n)), \qquad C = (c_1, \dots, c_n)$$

and

$$A_{ij} = 0 \quad \text{if } i \neq j \quad \text{and} \quad A_{ii} = a_i \quad \text{for } i, j = 1, \dots, n.$$

The following corollary concerns a refinement type equation of form (1.5) with a complete measure P and the functions A, F, C defined above.

Corollary 3.1. *Assume that $a_1, \dots, a_n$ are positive (resp. negative), $F_1, \dots, F_n$ are increasing (resp. decreasing) and*

$$-\infty < \int_\Omega \log \max_{k=1,\dots,n} |a_k(\omega)| P(d\omega) < 0.$$

Then the trivial function is the only L^1-solution $\Phi\colon \mathbb{R}^n \to \mathbb{R}$ of (1.5).

Proof. Suppose that $\Phi\colon \mathbb{R}^n \to \mathbb{R}$ is an L^1-solution of (1.5). Define $\varphi\colon \mathbb{R}^n \to \mathbb{R}$ by

$$\varphi(x) = \int_{U_x} \Phi(t)dt,$$

where $U_x = (-\infty, x_1) \times \cdots \times (-\infty, x_n)$ for $x \in \mathbb{R}^n$. Since

$$U_x = f^{-1}(\cdot, \omega)(U_{f(x,\omega)})$$

and the function "$\mathbb{R}^n \times \Omega \ni (x,\omega) \mapsto |\det A(\omega)F'(x)|\Phi(A(\omega)F(x) - C(\omega))$" is product measurable, it follows that

$$\varphi(x) = \int_\Omega \left(\int_{U_x} |\det A(\omega)F'(t)|\Phi(A(\omega)F(t) - C(\omega))dt \right) P(d\omega) =$$

$$= \int_\Omega \left(\int_{U_{f(x,\omega)}} \Phi(t)dt \right) P(d\omega) = \int_\Omega \varphi(A(\omega)F(x) - C(\omega))P(d\omega).$$

This means that φ is a bounded and uniformly continuous solution of (2.7). Moreover, all the assumptions of Corollary 2.8(i) are satisfied. Consequently, φ is constant and so Φ equals zero. □

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