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ON THE DIAMETER OF DOT-CRITICAL GRAPHS

Abstract. A graph G is k-dot-critical (totaly k-dot-critical) if G is dot-critical (totaly dot-critical) and the domination number is k. In the paper [T. Burtona, D. P. Sumner, Domination dot-critical graphs, Discrete Math, 306(2006), 11–18] the following question is posed: What are the best bounds for the diameter of a k-dot-critical graph and a totally k-dot-critical graph G with no critical vertices for $k \geq 4$? We find the best bound for the diameter of a k-dot-critical graph, where $k \in \{4, 5, 6\}$ and we give a family of k-dot-critical graphs (with no critical vertices) with sharp diameter $2k - 3$ for even $k \geq 4$.

Keywords: dot-critical graph, domination, diameter.

Mathematics Subject Classification: 05C69.

1. INTRODUCTION

Let $G = (V, E)$ be a graph. A set $S \subseteq V$ is a dominating set if every vertex of $V \setminus S$ is adjacent to some vertex of S. If S has the smallest possible cardinality of among all dominating sets of G, then S is called a minimum dominating set (abbreviated MDS) of G. The cardinality of any MDS of G is called the domination number of G and is denoted by $\gamma(G)$ [5] . More generally, we say that a set of vertices A dominates the set of vertices B if every vertex of $B \setminus A$ is adjacent to some vertex of A. Two graphs G_1 and G_2 are disjoint if they have no vertex in common and no vertex of G_1 is adjacent to any vertex of G_2. We denote the open neighborhood of a vertex v in G by $N_G(v)$ and its closed neighborhood by $N_G[v]$ (so we have $N_G[v] = N_G(v) \cup \{v\}$). We indicate the fact that the vertex v is adjacent to a vertex u by writing $v \leftrightarrow u$. We denote the edge with endpoints v and u by vu, the diameter of G by $d = d(G)$ and the length of the path with endpoints v and w with $d(v, w)$. Let A_i and A_j be the sets of vertices. We indicate the fact that every element of A_i is adjacent to every element of A_j by writing $A_i \leftrightarrow A_j$ and that the induced subgraph $\langle A_i \rangle$ is clique by writing ${A_i}^c$. If a property of graphs is worth studying, then it is almost certainly worthwhile to investigate those graphs that are extreme with respect to that property. But there may be many ways in which a graph can be extreme. In particular, for the domination number, there are a variety of extremal concepts that have been investigated. The two most studied are the edge-critical graphs introduced by Sumner and Blitch [6] and the vertex-critical graphs introduced by Brigham et al [1] . A graph G is edge-critical with respect to the domination number if for every two non-adjacent vertices v and u, $\gamma(G+vu) < \gamma(G)$. A vertex v of G is critical if $\gamma(G-v) < \gamma(G)$. A graph G is vertex-critical if every vertex of G is critical. We denote the set of critical vertices of G by V^-. A vertex v is stable if $\gamma(G-v) = \gamma(G)$. In [3,4] a new critical condition for the domination number has been introduced. A graph is domination dot-critical (hereafter, just dot-critical) if identifying any two adjacent vertices (i.e., contracting the edge comprising those vertices) results in a graph with smaller domination number. If identifying any two vertices of G causes the domination number to decrease, then we say that G is totally dot-critical. When we say that G is k-edge-critical, k-vertex-critical, k-dot-critical, or totally-k-dot-critical, we mean that it has the indicated property and that $\gamma(G) = k$. In the paper [4] T. Burton, and D.P. Sumner posed the question: What are the best bounds for the diameter of a k-dot-critical graph and a totally k-dot-critical graph G with no critical vertices for $k \geq 4$? We find the best bound for the diameter of a k-dot-critical graph, where $k \in \{4; 5; 6\}$, and we give a family of k-dot-critical graphs (with $V^- = \emptyset$) with sharp diameter $2k-3$ for even $k \geq 4$. We believe this bound can be $2k-1$ for odd $k \geq 5$. The paper ends with some open problems.

Note that, after we had sent the paper to Opuscula Mathematica for reviewing we became aware of the papers ([“Domination dot-critical graphs with no critical vertices”, Zhao Chengye, Yang Yuansheng, Sun Linlin, Discrete Math. Volume 308, Issue 15 (2008)] and [“On the diameter of a domination dot-critical graph”, Nader Jafari Rad, Discrete Applied Mathematics 157 (2009), 1647-1649]). These papers, especially the second one, have some similar results, though we are sure that the second mentioned paper has been prepared after ours.

The following facts are useful.

Lemma A ([4]) If G is any graph with $\gamma(G) = k \geq 2$, then G is dot-critical (resp. totally dot-critical) if and only if every two adjacent non-critical vertices (resp. any two non-critical vertices) belong to a common MDS.

Lemma B ([4]) Let G be a dot-critical graph, v and u be two vertices of G. If $N_G[v] \subseteq N_G[u]$, then $v \in V^-$.

2. EXAMPLES

We give some examples of dot-critical and totally dot-critical graphs:

1. P_{3k+1} is dot-critical.
2. C_{3k+1} is totally dot-critical.
3. Let K_t be a complete graph and $K_t \square K_t$ be the Cartesian product of K_t with itself (see Fig. 1). In [2] R.C. Brigham *et al.* have shown that the graph $K_t \square K_t$ is t-vertex critical for $t \geq 3$. Hence by Lemma A, $K_t \square K_t$, $(t \geq 3)$ is totally t-dot-critical, because the graph does not have any non-critical vertices.

4. The circulant graph $C_{12}\langle 1,4\rangle$ (see Fig. 2) is the graph with vertex set $\{v_0, v_1, \cdots, v_{11}\}$ and edge set $\{v_i v_{i+j(mod\ 12)} |\ i \in \{0, 1, \cdots, 11\}$ and $j \in \{1, 4\}\}$. The graph $G = C_{12}\langle 1,4\rangle$ has domination number 4, and $\{v_0, v_3, v_6, v_9\}$ is an MDS of G. The set of vertices $\{v_3, v_6, v_9\}$ dominates $G - \{v_0\}$ and G is vertex transitive, therefore G is critical. By Lemma A, G is totally 4-dot-critical.

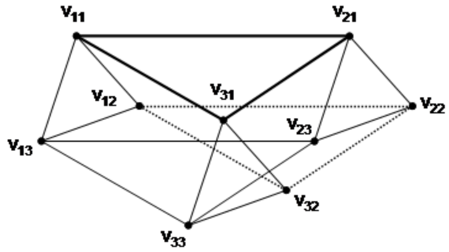

Fig. 1. The graph $K_3 \square K_3$

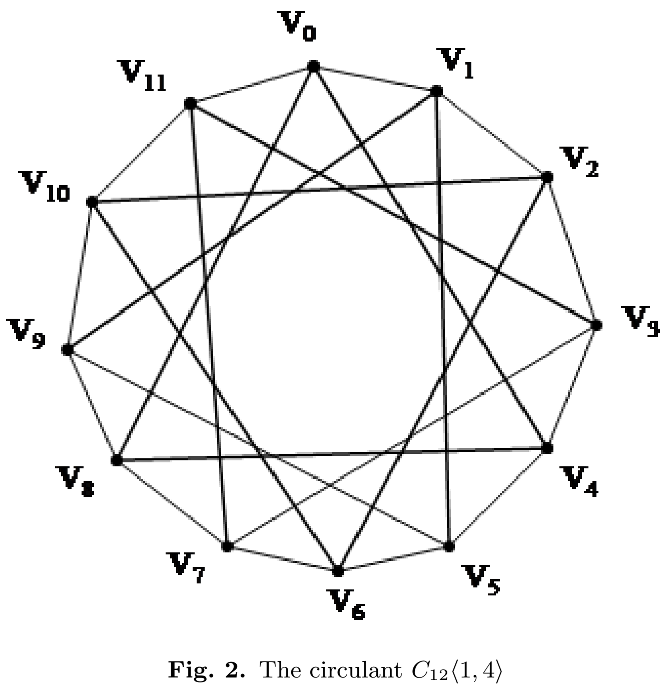

Fig. 2. The circulant $C_{12}\langle 1,4\rangle$

5. The following figures (see Fig. 3) denote some examples of dot-critical graphs with no critical vertices. Each of them is constructed as follows. Let $G = (V, E)$ be a

graph where
$V = \{u_1, u_2, \cdots, u_n, u_{11}, u_{12}, \cdots, u_{1k}, u_{21}, u_{22}, \cdots, u_{2k}, \cdots, u_{n1}, u_{n2}, \cdots, u_{nk}\}$
and
$E = \{u_i u_{i-1}, u_i u_{i+1}, u_i u_{(i-1)j}, u_i u_{(i+1)j}, u_{il} u_{(i-1)j}, u_{il} u_{(i+1)j} | 1 \leq i(\text{mod}, n) \leq n$ and $1 \leq j, l \leq k$, where $k \geq 2\}$.
For $n \in \{4, 5\}$ and $k = 2$, see Fig. 3.

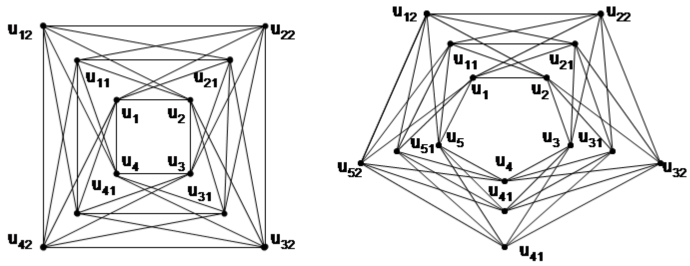

Fig 3.

3. MAIN RESULTS

The diameters of k-dot-critical graphs ($k \geq 4$) with no critical vertices are studied.

Theorem 3.1. *A connected* 4*-dot-critical graph with* $V^- = \emptyset$*, has diameter of at most five.*

Proof. Let G be a connected 4-dot-critical graph with $V^- = \emptyset$ and diameter d. Let $v \in V(G)$ be a vertex so that there is a vertex $w \in V$ and $d(v, w) = d$. Let $A_i = \{x \in V(G) | d(v, x) = i\}$ for $0 \leq i \leq d$. Then $A_i \neq \emptyset$ and $A_0 = \{v\}$. For any $x \in A_1$, $v \leftrightarrow x$. The assumption says that any MDS has 4 vertices and according to Lemma A any two adjacent vertices belong to an MDS. Let $x \in A_1$, and suppose there is an MDS containing the vertices v and x. Since these two vertices dominate at most $A_0 \cup A_1 \cup A_2$, so two other vertices of the MDS, s_3 and s_4 say, dominate $\bigcup_{i=3}^{d} A_i$. Each s_i can dominate at most three of A_i's for $3 \leq i \leq d$. Hence it results that $d \leq 8$. The following facts show that $d \neq 6, 7, 8$.

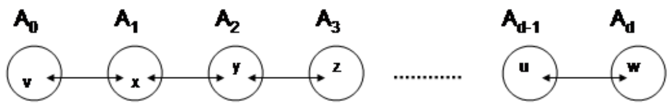

Fig 4.

Fact 1.1. Suppose $d = 8$. Let $y \in A_2$, $z \in A_3$ and $y \leftrightarrow z$. There exists an MDS of G containing both of y and z. Thus two other vertices of the MDS satisfy

$\{s_3, s_4\} \subseteq \bigcup_{\substack{i=0 \\ i\neq 2,3}}^{8} A_i$. Let $s_3 \in A_0 \cup A_1$ and suppose that $\bigcup_{i=4}^{8} A_i$ contains the vertex s_4 which dominates at most three of A_i's. This implies that $s_4 \in A_6$ and then A_8 is not dominated by the MDS. Hence $d \leq 7$.

Fact 1.2. Suppose $d = 7$. If we consider the MDS of Fact 1.1, then for any $w \in A_7$, $N_G[w] \subseteq N_G[s_4]$ and so $w \in V^-$ contradicts Lemma B. Thus $d \leq 6$.

Fact 1.3. Suppose $d = 6$. There are some cases.
Case 1.1. Let MDS be the set $S = \{y, z, s_3, s_4\}$ where $y \in A_2$ and $z \in A_3$, then $\{s_3, s_4\} \subseteq \bigcup_{\substack{i=0 \\ i\neq 2,3}}^{6} A_i$. Suppose $s_3 \in A_0 \cup A_1$. Then the vertex s_4 dominates $A_5 \cup A_6$. If $s_4 \in A_5$, then for every $w \in A_6$, $N_G[w] \subseteq N_G[s_4]$, which implies $w \in V^-$, a contradiction. Hence $s_4 \in A_6$. We claim that $A_6 = \{s_4\}$. Let $u \in A_6 \setminus \{s_4\}$, then $N_G[u] \subseteq N_G[s_4]$ and $u \in V^-$, a contradiction, so $A_6 = \{s_4\}$ and $A_5 \leftrightarrow A_6$. Thus the only element of S that can dominate A_4 is $z \in A_3$ and since z has been chosen arbitrary, it means that $A_4 \leftrightarrow A_3$.
Case 1.2. Let MDS be the set $S = \{z, u, s_3, s_4\}$ where $z \in A_3$ and $u \in A_4$; then $s_3 \in A_5 \cup A_6$ and by Lemma B $s_4 = v \in A_0$. Thus the only element of S that can dominate A_2 is $z \in A_3$ and since z has been chosen arbitrary, it means that $A_2 \leftrightarrow A_3$.
Case 1.3. Let MDS be the set $S = \{v, x, s_3, s_4\}$ where $v \in A_0$ and $x \in A_1$, thus $\bigcup_{i=2}^{6} A_i$ contains the vertices of the set $\{s_3, s_4\}$. There are some subcases.
Subcase 1.3.1. If $s_3 \in A_2 \cup A_3$ and $s_4 \in A_5$, then for any $w \in A_6$, $N_G[w] \subseteq N_G[s_4]$ and so $w \in V^-$ contradicts Lemma B.
Subcase 1.3.2. Suppose that $s_3 \in A_3$, $s_4 \in A_6$, A_6 has only one element by Case 1.1 and A_5 is dominated by s_4. It is clear that any vertex $z \in A_3$ is dominated by $s_3 \in A_3$ and since s_3 is arbitrary, then A_3 is clique. This with together Cases 1.1 and 1.2 implies that $\bigcup_{i=2}^{4} A_i$ is dominated by s_3. Since the vertex v dominates A_1, hence $S = \{v, s_3, s_4\}$ is an MDS, a contradiction.
Subcase 1.3.3. Suppose that $s_3 \in A_4$, $s_4 \in A_5 \cup A_6$, so the only element of MDS that can dominate A_2 is $x \in A_1$ and since x is an arbitrary, it means that $A_1 \leftrightarrow A_2$.
Case 1.4. Let MDS be the set $S = \{u, w, s_3, s_4\}$ where $u \in A_5$ and $w \in A_6$. There are some subcases.
Subcase 1.4.1. Suppose $s_3 \in A_4$, so s_4 must be in A_1 and then $N_G[v] \subseteq N_G[s_4]$ and $v \in V^-$, a contradiction.
Subcase 1.4.2. Suppose $s_3 \in A_3$ and $s_4 \in A_0 \cup A_1$, then A_3 is clique and using Cases 1.1, 1.2 and 1.3 implies $A_2 \leftrightarrow A_3^c \leftrightarrow A_4$. Now $\{s_4, s_3, w\}$ is an MDS of G, a contradiction.
Subcase 1.4.3. Suppose $s_3 \in A_2$ and $s_4 \in A_0 \cup A_1$, then $u \leftrightarrow A_4$. The vertex $u \in A_5$ is arbitrary, so $A_4 \leftrightarrow A_5$.

Using Cases 1.1–1.4, we have a contradiction or $A_0 \leftrightarrow A_1 \leftrightarrow A_2 \leftrightarrow A_3 \leftrightarrow A_4 \leftrightarrow A_5 \leftrightarrow A_6$. We show that the last relation leads to a contradiction. For this, let

$S = \{y, z, w\}$ where $y \in A_2$, $z \in A_3$ and $w \in A_6$, then $\{y, z\}$ dominates $\bigcup_{i=1}^{4} A_i$ and w dominates $A_5 \cup A_6$. Hence for $v \in A_0$ the set $S = \{y, z, w\}$ is an MDS for $G - v$ and $v \in V^-$, a contradiction. These contradictions show that $d \leq 5$. □

The bound on a diameter in Theorem 1 is sharp, see the following example.
Example 1. The graph G in Figure 5 is a 4-dot-critical with $V^- = \emptyset$ and diameter $d = 5$.

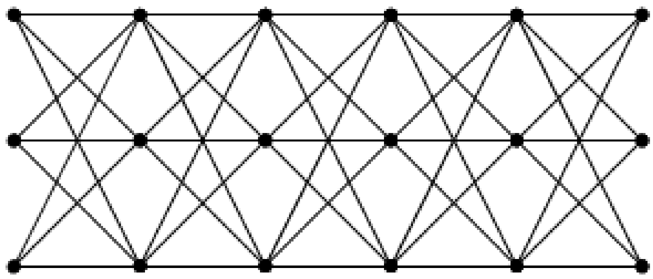

Fig 5.

Theorem 3.2. *A connected* 5*-dot-critical graph with* $V^- = \emptyset$ *has a diameter of at most seven.*

Proof. Let G be a connected 5-dot-critical graph with $V^- = \emptyset$ and diameter d. Let v and $w \in V(G)$ be such that $d(v, w) = d$. By Theorem 1, $A_i \neq \emptyset$ and $A_0 = \{v\}$. For any $x \in A_1$, $v \leftrightarrow x$ and there is an MDS of G containing both v and x so also $\{v, x, s_3, s_4, s_5\}$. Thus, $\bigcup_{i=3}^{d} A_d$ is dominated by three vertices $\{s_3, s_4, s_5\}$. Hence $d \leq 11$. Through the following facts we show that $d \neq 8, 9, 10, 11$.
Fact 2.1. Suppose $d = 11$. For $u \in A_4$ and $w \in A_5$ there is an MDS of G containing both u and w. If $A_0 \cup A_1 \cup A_2$ is dominated by one vertex of this MDS, s_3 say, then $s_3 \in A_1$, and $N_G[v] \subseteq N_G[s_3]$ and so $v \in V^-$, a contradiction. Hence $\bigcup_{i=0}^{3} A_i$ is dominated by two vertices $\{s_3, s_4\}$. Thus $\bigcup_{i=7}^{d} A_i$ is dominated by one vertex s_5. Hence $d \leq 9$.
Fact 2.2. Suppose that $d = 9$ and $S = \{u, w, s_3, s_4, s_5\}$ is an MDS where $u \in A_4$ and $w \in A_5$. In the same way as in Fact 2.1, $\{s_3, s_4\}$ dominates $\bigcup_{i=0}^{3} A_i$ and $s_5 \in A_8$. Then for any $u \in A_9$, $N_G[u] \subseteq N_G[s_5]$ and $u \in V^-$, a contradiction. Hence $d \leq 8$.
Fact 2.3. Suppose that $d = 8$ and MDS is the set $S = \{u, w, s_3, s_4, s_5\}$. There are some cases.
Case 2.1. Let $u \in A_4$ and $w \in A_5$, then $\{s_3, s_4\} \subseteq \bigcup_{i=0}^{3} A_i$. If $s_5 \in A_7$, then for any $u \in A_8$, $N_G[u] \subseteq N_G[s_5]$ and $u \in V^-$, a contradiction. If $s_5 \in A_8$ and there is a vertex

$(s_5 \neq) y \in A_8$, then $N_G[y] \subseteq N_G[s_5]$, so $y \in V^-$, a contradiction. Hence $A_8 = \{c\}$, $A_7 \leftrightarrow A_8$ and $A_5 \leftrightarrow A_6$ because $w \in A_5$ is an arbitrary vertex that dominates A_6.

Case 2.2. Let $u \in A_3$, $w \in A_4$, then $\{s_3, s_4\} \subseteq \bigcup_{i=5}^{8} A_i$ and $s_5 = v \in A_0$. So u dominates A_2 and $A_2 \leftrightarrow A_3$, because $u \in A_3$ is an arbitrary vertex that dominates A_2.

Case 2.3. Let $u \in A_2$ and $w \in A_3$, then $s_3 \in A_0 \cup A_1$ and $\bigcup_{i=5}^{8} A_i$ is dominated by two vertices.

Case 2.4. Let $u \in A_5$ and $w \in A_6$, then $s_5 \in A_7 \cup A_8$ and $\bigcup_{i=0}^{3} A_i$ is dominated by two vertices.

Case 2.5. Let $u(= v) \in A_0$ and $w \in A_1$, then $\{s_3, s_4, s_5\} \subseteq \bigcup_{i=2}^{8} A_i$. The following subcases may take place.

Subcase 2.5.1. Let $s_3 \in A_2$, $s_4 \in A_5$ and $s_5 \in A_8$, then $\bigcup_{i=4}^{8} A_i$ is dominated by two vertices. Now using Case 2.4 one can conclude that $\bigcup_{i=0}^{8} A_i$ is dominated by 4 vertices, a contradiction.

Subcase 2.5.2. Let $s_3 \in A_3$, $s_4 \in A_5$ and $s_5 = c \in A_8$. These assumptions and Case 2.2 imply that the set $\{s_3, s_4, s_5\}$ dominates $\bigcup_{i=2}^{8} A_i$. Therefore $S = \{u, s_3, s_4, s_5\}$ is an MDS of G, a contradiction.

Subcase 2.5.3. Let $s_3 \in A_3$, $s_4 \in A_6$ and $s_5 = c \in A_8$. Then similarly as in Subcase 2.5.2, $S = \{u, s_3, s_4, s_5\}$ is an MDS of G, a contradiction.

Subcase 2.5.4. Let $\{s_3, s_4, s_5\} \subseteq \bigcup_{i=4}^{8} A_i$, then w dominates A_2. Since w is an arbitrary vertex of A_1, hence $A_1 \leftrightarrow A_2$. Now we immediately flash back to Case 2.3 where the MDS is $S = \{u', w', s'_3, s'_4, s'_5\}$ with $u' \in A_2$, $w' \in A_3$ and $s'_3 \in A_0$, and one can see that the set $S = \{u', w', s'_4, s'_5\}$ dominates $\bigcup_{i=1}^{8} A_i$. Hence $\gamma(G - v) = 4$ for $v \in A_0 = \{v\}$, a contradiction. Since there is an MDS with two adjacent vertices $u = v \in A_0$ and $w \in A_1$ and when $d = 8$ this anyway leads to a contradiction. Thus $d \leq 7$. □

Problem 1. Is there a connected 5-dot-critical graph with $V^- = \emptyset$ and $d = 7$?

Theorem 3.3. *A connected* 6*-dot-critical graph with* $V^- = \emptyset$ *has a diameter of at most nine.*

Proof. Let G be a connected 6-dot-critical graph with $V^- = \emptyset$ and diameter d. Let $v, w \in V(G)$ be such that $d(v, w) = d$. Let $A_i = \{x \in V(G) | d(v, x) = i\}$ for $0 \leq i \leq d$. So $A_0 = \{v\}$ and for $x \in A_1$, $v \leftrightarrow x$. By Lemma A, there exists an MDS of G containing both v and x, like $\{v, x, s_3, s_4, s_5, s_6\}$. Thus $\bigcup_{i=3}^{d} A_i$ are dominated by four vertices $\{s_3, s_4, s_5, s_6\}$. Hence $d \leq 14$. Through the following facts we show that $d \neq 10, 11, 12, 13, 14$.

Fact 3.1. Suppose that $d = 14$, $u \leftrightarrow w$ and $S = \{u, w, s_3, s_4, s_5, s_6\}$ is an MDS of G. Let $u \in A_4$ and $w \in A_5$. Then $A_0 \cup A_1 \cup A_2$ is not dominated by one vertex $s_3 \in A_1$, because we will have $N_G[v] \subseteq N_G[s_3]$ for $v \in A_0$ and then $v \in V^-$, a contradiction. Hence $\{s_3, s_4\} \subseteq \bigcup_{i=0}^{4} A_i$ and $\bigcup_{i=7}^{d} A_i$ is dominated by two vertices of $\{s_5, s_6\}$, a contradiction. Hence $d \leq 12$.

Fact 3.2. Suppose that $d = 12$ and $S = \{u, w, s_3, s_4, s_5, s_6\}$ is an MDS of G where $u \in A_4$ and $w \in A_5$. Then $\{s_3, s_4\} \subseteq \bigcup_{i=0}^{4} A_i$, $s_5 \in A_8$ and $s_6 \in A_{11}$. If $x \in A_{12}$, then $N_G[x] \subseteq N_G[s_6]$ and so $x \in V^-$, a contradiction, Hence $d \leq 11$.

Fact 3.3. Suppose that $d = 11$ and $\{u, w, s_3, s_4, s_5, s_6\}$ is an MDS of G. There are some cases.

Case 3.1. Let $u \in A_4$ and $w \in A_5$; then $\{s_3, s_4\} \subseteq \bigcup_{i=0}^{3} A_i$, $s_5 \in A_8$ and $s_6 \in A_{11}$. If there is a vertex $y \in A_{11} - s_6$, then $N_G[y] \subseteq N_G[s_6]$, and so $y \in V^-$. Thus $A_{11} = \{c\}$. Therefore we have $A_5 \leftrightarrow A_6$, $A_{11} \leftrightarrow A_{10}$ and $\bigcup_{i=7}^{11} A_i$ is dominated by two vertices.

Case 3.2. Let $u \in A_6$ and $w \in A_7$ then $\{s_3, s_4\} \subseteq \bigcup_{i=8}^{11} A_i$ and $s_5 \in A_0$ and $s_6 \in A_3$. Then $\bigcup_{i=0}^{4} A_i$ is dominated by two vertices.

Case 3.3. Let $u \in A_2$, $w \in A_3$ then $s_3 \in A_0 \cup A_1$, so we consider the following subcases.

Subcase 3.3.1. If $s_4 \in A_5$, $s_5 \in A_7$, then $s_6 \in A_{10}$ and $c \in A_{11}$ would be a critical vertex, a contradiction.

Subcase 3.3.2. $s_4 \in A_5$, $s_5 \in A_8$ and $s_6 \in A_{11} = \{c\}$.

Subcase 3.3.3. $s_4 \in A_6$, $s_5 \in A_8$ and $s_6 \in A_{11} = \{c\}$.

Subcase 3.3.4. $s_4 \in A_6$, $s_5 \in A_9$ and $s_6 \in A_{10}$.

Subcase 3.3.5. $s_4 \in A_6$, $s_5 \in A_9$ and $s_6 \in A_{11} = \{c\}$.

These Subcases result in a contradiction or $A_5 \cup A_6$ is dominated by one vertex s_4. Now combining Cases 3.1, 3.2 and 3.3 implies that the graph G is dominated by 5 vertices, a contradiction. Hence $d \leq 10$.

Fact 3.4. Suppose that $d = 10$ and $\{u, w, s_3, s_4, s_5, s_6\}$ is an MDS of G. There are some cases.

Case 3.4. Let $w \in A_7$ and $u \in A_8$, then $s_3 \in A_9 \cup A_{10}$ and $\bigcup_{i=0}^{5} A_i$ is dominated by the set $\{s_4, s_5, s_6\}$.

Case 3.5. Let $w \in A_3$ and $u \in A_4$, then we have the following subcases.

Subcase 3.5.1. If $A_0 \cup A_1 \cup A_2$ contains 2 vertices of the MDS, then $\bigcup_{i=6}^{10} A_i$ is dominated by 2 vertices. This result combined with Case 3.4 yields that G has an MDS with size 5, a contradiction.

Subcase 3.5.2. Suppose that $A_0 \cup A_1 \cup A_2$ contains one vertex s_3 of the MDS.
If $s_3 \in A_1$, then $N_G[v] \subseteq N_G[s_3]$ and so $v \in V^-$, a contradiction.
If $s_3 = v \in A_0$, then $w \leftrightarrow A_2$. Since w is an arbitrary vertex in A_3, then $A_2 \leftrightarrow A_3$

Case 3.6. Let $u \in A_5$ and $w \in A_6$, then $\bigcup_{i=0}^{3} A_i$ is dominated by 2 vertices and so does $\bigcup_{i=8}^{10} A_i$ by 2 other vertices of the MDS.

Case 3.7. Let $u = v \in A_0$ and $w \in A_1$, then $A_2 \cup A_3 \cup A_4$ contains one vertex s_3 of MDS (otherwise $\bigcup_{i=6}^{10} A_i$ is dominated by 2 vertices and by Case 3.4 we have a contradiction). We consider the following subcases.

Subcase 3.7.1. Suppose that $s_3 \in A_2$, so $\bigcup_{i=4}^{10} A_i$ is dominated by 3 vertices. This result combined with Case 3.6 yields that G has an MDS with size 5, a contradiction.

Subcase 3.7.2. Suppose that $s_3 \in A_3$. Since $A_2 \leftrightarrow A_3$ and $A_0 \leftrightarrow A_1$, then one can replace $\{u, w, s_3\}$ with $\{u, s_3\}$. Hence G has an MDS with size 5, a contradiction.

Subcase 3.7.3. Suppose that $s_3 \in A_4$. Then $s_3 \leftrightarrow A_3$ and $w \leftrightarrow A_2$. Since w is an arbitrary vertex, then $A_1 \leftrightarrow A_2$.

Case 3.8. Let $u \in A_2$ and $w \in A_3$, then u dominates A_1 (Subcase 3.7.3) and one can choose $s_3 = v \in A_0$. Thus $\gamma(G - v) = 5$ and $v \in V^-$, a contradiction. These Cases show that there a 6-dot critical graph with diameter 10 does not exist. Therefore $d \leq 9$ and the proof is completed. □

The bound on diameter in Theorem 3 is sharp, see the following example.

Example 2. The graph G in Figure 6 is a 6-dot-critical with no critical vertices and diameter $d = 9$.

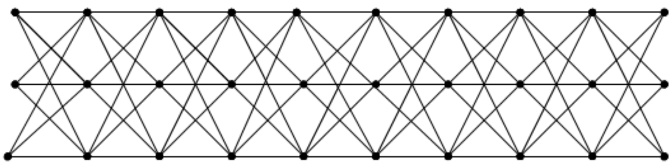

Fig 6.

We will show a family of dot-critical graphs with sharp diameter, see below.

Proposition 3.4. *Let $n \geq 4$ be an even number. There is a family of n-dot-critical graph G with $V^- = \emptyset$ and diameter $d = 2n - 3$.*

Proof. For $n = 4$ and $n = 6$ Figures 5 and 6 show the necessary result. Let G be a graph with vertex set $\{v_{i1}, v_{i2}, \cdots, v_{i(4m-2)} | \ 1 \leq i \leq k \text{ and } k \geq 3\}$ and edge set $\{v_{ij}v_{l(j+1)} | \ 1 \leq i, l \leq k \text{ and } 1 \leq j \leq 4m - 3\}$ (see Fig. 7) where $m \geq 4$. If $n = 2m$, then the set $S = \{4t + 2, 4t + 3 | \ 0 \leq t \leq m - 2\} \cup \{4m - 3, 4m - 2\}$ is an MDS of G. It is easily seen that G is dot-critical, $V^- = \emptyset$ and its diameter is $2m - 3$. The different values of k give us a family of $n = 2m$-dot-critical graphs with $V^- = \emptyset$ and diameter $d = 4m - 3$. □

Example 3. The graph below (Fig. 7) is a connected 8-dot-critical graph with $V^- = \emptyset$ and diameter $d = 13$.

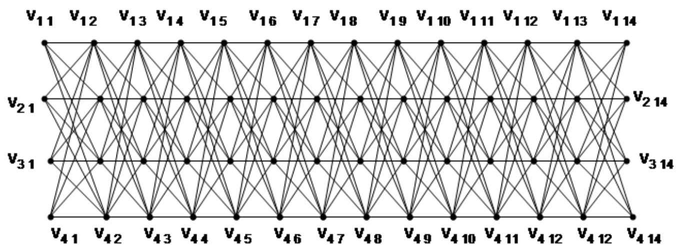

Fig 7.

Problem 2. Is there a connected $n = 2m + 1$-dot-critical graph with $V^- = \emptyset$ and diameter $d = 2n - 3$?

The studied results make us believe that for any $n \geq 7$, using the method of the proofs of Theorems 1-3 with more cases and Proposition 3.4 will give us the similar results.
Our belief is posed as a conjecture.

Conjecture. For $n \geq 9$, a connected n-dot-critical graph with $V^- = \emptyset$ has a diameter of at most $2n - 3$.

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