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## SMOOTHED ESTIMATOR OF THE PERIODIC HAZARD FUNCTION

**Abstract.** A smoothed estimator of the periodic hazard function is considered and its asymptotic probability distribution and bootstrap simultaneous confidence intervals are derived. Moreover, consistency of the bootstrap method is proved and some applications of the developed theory are presented. The bootstrap method is based on the phase-consistent resampling scheme developed in Dudek and Leśkow [6].

**Keywords:** bootstrap, consistency, multiplicative intensity model, periodic hazard function.

**Mathematics Subject Classification:** 62G09, 62G07, 60G55.

### 1. INTRODUCTION

Estimation of hazard function is very important in many research fields like biostatistics or telecommunication. The case of a periodic hazard function has been considered in [11], where it has been shown that the asymptotic distribution of the developed periodic histogram sieve estimator is normal. Dudek and Leśkow have shown in [6] that this result remains valid under slightly weaker assumptions and, moreover, they constructed bootstrap simultaneous confidence intervals for the hazard function. The bootstrap estimator was based on a new phase-consistent resampling scheme (*PCRS*), retaining the temporal order and henceforth applicable to problems with a periodic structure. Let us note that other existing resampling schemes considered in [2]–[5] are applicable only in problems with a stationary structure.

The histogram sieve estimator introduced in [11] is piecewise constant, while in applications rather smoothed versions of the estimator are preferable. Therefore Leśkow considered in [12] a kernel smoothed version of the periodic histogram sieve estimator. Unfortunately, we think that the proof of the theorem in which the asymptotic distribution was established in deriving its asymptotic distribution is incomplete. There is a mistake where the variance is calculated. In the present paper we rectify this but

under different conditions and, following [6], we use the *PCRS* to construct bootstrap simultaneous confidence intervals for the smoothed version of the hazard function.

The paper is organized in the following way. Section 2 contains the formulation of the problem and the assumptions that are essential for our results. The asymptotic distribution of the considered estimator is established in Section 3. In Section 4 the consistency of *PCRS* is proved. Some concluding remarks are placed in Section 5.

## 2. PROBLEM FORMULATION

Let  $(\Omega, \mathcal{F}, P)$  be a probability space with a filtration  $\mathcal{F}_t$ . We observe a counting process  $\{X(t), t \in \mathcal{T}\}$ , where  $\mathcal{T} = [0, T]$ . We consider the Nelson-Aalen model in which the stochastic intensity of  $X(t)$  is of the form:  $\lambda(t) = \lambda_0(t)Y(t)$ . The function  $\lambda_0(t)$  is deterministic and nonnegative and it is called the hazard function. The stochastic process  $Y(t)$  is nonnegative, left-continuous and adapted (for more details see [2], chapter II).

The multiplicative intensity model is very popular especially in biomedical settings because of its interpretation. One may consider  $Y(t)$  as the number at risk at time  $t$  (that can be number of patients after some medical treatment) and  $\lambda_0(t)$  for example as the intensity of death.

In the sequel it will be assumed that  $\lambda_0(t)$  is a periodic function with the period  $P$  and  $Y(t)$  is periodically correlated which means that it has periodic mean and covariance functions. Such model will be called the periodic multiplicative intensity model. It is considered also in [6], where applications to some telecommunication traffic real data example are shown. Leśkow in [11] introduced the histogram maximum likelihood estimator of the periodic function. The idea of its construction is to split a single realization of the process  $X$  to obtain the family of counting processes  $\{X_k(t) : t \in [0, P], k = 1, 2, \dots\}$ :

$$\begin{aligned} X_k(t) &= X(t + P(k-1)) - X(P(k-1)), \\ Y_k(t) &= Y(t + P(k-1)), \end{aligned}$$

where  $t \in [0, P]$  and  $k = 1, 2, \dots$

The stochastic intensity of the process  $X_k(t)$  is of the form

$$\lambda_k(t) = \lambda_0(t)Y_k(t), \quad t \in [0, P].$$

The estimator of  $\lambda_0(t)$  is defined as follows

$$\hat{\lambda}_n(t) = \sum_{l=1}^{m_n} \frac{\sum_{k=1}^n X_k(A_{l,m_n})}{\sum_{k=1}^n \int_{A_{l,m_n}} Y_k(u) du} 1_{A_{l,m_n}}(t), \quad t \in [0, P], \quad (2.1)$$

where  $A_{l,m_n} = ((l-1)P/m_n, lP/m_n]$  ( $l = 1, \dots, m_n$ ) is the  $l$ -th subinterval of  $[0, P]$  and, with some abuse of notation, we write  $X_k(A_{l,m_n})$  for  $X_k(lP/m_n) - X_k((l-1)P/m_n)$ . The same convention will be used throughout the paper for all other processes, especially those arising from the Doob-Meyer decomposition. The estimator

$\hat{\lambda}_n$  is defined on the set  $D_n = \{\sum_{k=1}^n \int_{A_{l,m_n}} Y_k(u) du > 0\}$ . When the denominator of (2.1) is equal to zero we put  $\hat{\lambda}_n(t) = 0$ .

In papers [11] and [6] the asymptotic distribution of  $\hat{\lambda}_n(t)$  is determined. In [6] this result is obtained under less restrictive assumptions and additionally the multi-dimensional case is considered.

For kernels  $K$  with support  $[-1, 1]$  it is natural to define a smoothed version of the estimator by

$$\hat{\eta}_n(t) = \frac{1}{Pb_n} \int_{-\infty}^{\infty} K\left(\frac{t-s}{Pb_n}\right) \hat{\lambda}_n(s) ds = \frac{1}{Pb_n} \int_{t-Pb_n}^{t+Pb_n} K\left(\frac{t-s}{Pb_n}\right) \hat{\lambda}_n(s) ds, \quad (2.2)$$

where  $\hat{\lambda}_n(s) = 0$  for  $s \notin [0, P]$ , and hence for  $t \in [0, P]$  it can be written as

$$\hat{\eta}_n(t) = \frac{1}{Pb_n} \int_0^P K\left(\frac{t-s}{Pb_n}\right) \hat{\lambda}_n(s) ds. \quad (2.3)$$

We assume that the kernel function  $K$  with support  $[-1, 1]$  is nonnegative, bounded and fulfills the Lipschitz condition. Additionally  $\int_{-1}^1 K(u) du = 1$  and  $b_n \rightarrow 0$  as  $n \rightarrow \infty$ .

Moreover, in the sequel we assume that the following conditions hold:

- A1** Process  $Y$  is periodically correlated with period  $P$  and is almost surely bounded away from zero:  $Y(t, \omega) \geq \delta > 0$  and  $\delta$  does not depend on  $t$  and  $\omega$ .  
**A2** The process  $Y$  is uniformly bounded:

$$Y(t, \omega) < C_0$$

and  $C_0$  does not depend on  $t$  and  $\omega$ .

- A3** Process  $Y$  is  $\alpha$ -mixing with  $\alpha(k) = o(k^{-2})$ .  
**A4** The rate of the growth of the sieve is  $m_n \propto \sqrt{n}$ .  
**A5** The hazard function  $\lambda_0$  is periodic (the length of the period is  $P$ ).  
**A6** The expected value of the process  $Y$  and the hazard function  $\lambda_0$  fulfill the Lipschitz condition on  $[0, P]$  i.e. there exist constants  $C_1$  and  $C_2$  such that for any  $s, t \in [0, P]$

$$\begin{aligned} |EY(s) - EY(t)| &\leq C_1 |s - t|, \\ |\lambda_0(s) - \lambda_0(t)| &\leq C_2 |s - t|. \end{aligned}$$

- A7**  $b_n \propto n^{-\nu}$ , where  $\nu \in (5/12, 1/2)$ .

Symbol  $\propto$  denotes proportionality. By  $a_n \propto b_n$  we mean that there exists a positive constant  $D_0$  such that  $a_n/b_n = D_0$ . Assumptions **A2**–**A6** are the same as in [6]. In **A1** instead of  $EY(t)$  the process  $Y$  is bounded away from zero. Nonnegativity is not a

very restrictive condition. It appears also in the fundamental work of Aalen [1] in the context of the integrated hazard function estimation. It means that our group at risk is always nonempty. On the other hand, **A2** denotes boundedness of the group at risk. In the real data applications, where  $Y(t)$  is e.g. a group of patients at time  $t$  after some medical treatment or the number of working computers (see telecommunication example in [6]), this assumption is always fulfilled. Instead of  $\phi$ -mixing condition required in papers [11] and [12],  $\alpha$ -mixing is considered. Moreover,  $L^2$  continuity of  $Y$  process and periodicity of its distribution is replaced by assumption **A6**, which is easier to verify.

In the next sections we establish the asymptotic distribution of  $\hat{\eta}_n(t)$ . Moreover, we construct its bootstrap version  $\hat{\eta}_n^*(t)$  using the algorithm proposed in [6]. We show the consistency of this bootstrap method in one and multidimensional case.

### 3. ASYMPTOTIC RESULTS

In this Section we show that the asymptotic distribution of the estimator  $\hat{\eta}_n(t)$  is normal. Although similar result first appeared in [12], the proof given there does not seem complete. Especially,  $b_n = m_n^{-1}$  was assumed in [12], while we have to assume  $b_n$  to approach zero slower than  $m_n^{-1}$ . In effect, in contrary to [12], our kernel smoothes over an increasing number of histogram bins. The asymptotic variance in our result is equal to that proposed in [12]. It can be shown, however, that the sequence defined in [12] does not converge to this quantity.

**Theorem 3.1.** *Under **A1–A7**, for any  $t \in (0, P)$*

$$\sqrt{nb_n}(\hat{\eta}_n(t) - \lambda_0(t)) \xrightarrow{d} N(0, \sigma^2(t)), \quad (3.1)$$

where

$$\sigma^2(t) = \frac{\lambda_0(t)}{EY(t)} \int_{-1}^1 K^2(u) du. \quad (3.2)$$

Theorem 3.1 provides a way to construct the pointwise confidence intervals for the hazard function. Moreover, this result is the first step to get the simultaneous confidence intervals, which are much more important in applications (see [7]).

Before we present the proof of Theorem 3.1 we introduce the Doob-Meyer decomposition for counting processes, which may be found e.g. in [2] pp. 66–67.

For the counting process  $X$  the Doob-Meyer decomposition states the existence of a cadlag nondecreasing predictable process  $\Lambda$  such that

$$M = X - \Lambda \quad (3.3)$$

is a uniformly integrable martingale, zero at time zero. The process  $\Lambda$  is called the compensator of  $X$ .

In the multiplicative intensity model the compensator  $\Lambda$  is of the form (see [2] p. 177)

$$\Lambda(t) = \int_0^t \lambda_0(s)Y(s)ds. \quad (3.4)$$

In the following we denote by  $\Lambda_k(t)$  and  $M_k(t)$  the compensator and the related martingale of the counting process  $X_k(t)$ , respectively. Also, without loss of generality, we assume  $P = 1$ .

The following three lemmas will be essential for showing (3.1).

**Lemma 3.2.** *For any  $t \in (0, 1)$  and for  $n$  large enough we have*

$$\sum_{l=1}^{m_n} \frac{1}{b_n} \int_{A_{l,m_n}} K\left(\frac{t-s}{b_n}\right) ds = 1, \quad (3.5)$$

$$\sum_{l=1}^{m_n} \left( \frac{1}{b_n} \int_{A_{l,m_n}} K\left(\frac{t-s}{b_n}\right) ds \right)^2 \leq \frac{G}{b_n m_n}, \quad (3.6)$$

where  $G$  is a nonnegative positive constant independent of  $n$ .

*Proof.* The left-hand side of (3.5) can be rewritten as follows

$$\frac{1}{b_n} \int_0^1 K\left(\frac{t-s}{b_n}\right) ds = \frac{1}{b_n} \int_{t-bn}^{t+bn} K\left(\frac{t-s}{b_n}\right) ds.$$

Putting  $u = (t-s)/b_n$  we get  $\int_{-1}^1 K(u) du$ , which is equal to 1.

Inequality (3.6) is a straightforward consequence of nonnegativity and boundedness of the function  $K$  and the fact that the number of nonzero summands in the sum in question is  $2b_n m_n$  (the length of the integration interval is  $2b_n$  and it contains  $2b_n m_n$  intervals of the length  $1/m_n$ ).  $\square$

**Lemma 3.3.** *Define  $L_n(t)$  as follows*

$$L_n(t) = \frac{1}{EY(t)} \sum_{l=1}^{m_n} W_{l,n} \left( \frac{1}{b_n} \int_{A_{l,m_n}} K\left(\frac{t-s}{b_n}\right) ds \right), \quad (3.7)$$

where  $W_{l,n} = m_n/n \sum_{k=1}^n X_k(A_{l,m_n})$ .

Under **A1–A7**,  $L_n(t)$  and  $\hat{\eta}_n(t)$  are asymptotically equivalent, i.e.

$$\sqrt{nb_n} (L_n(t) - \hat{\eta}_n(t)) \xrightarrow{p} 0$$

for each  $t \in (0, 1)$ .

*Proof.* By  $U_{l,n}$  we denote the sum  $m_n/n \sum_{k=1}^n \int_{A_{l,m_n}} Y_k(u) du$ .

We show that

$$(nb_n)^{1/2} \sqrt{E|U_{l,n} - EY(t)|^2} \tag{3.8}$$

is bounded from above by a term which tends to zero as  $n \rightarrow \infty$  and is independent of  $l$  such that  $A_{l,m_n}$  is included in the interval  $(t - b_n, t + b_n)$ .

We have

$$\begin{aligned} \left[ E \left| (nb_n)^{1/2} (U_{l,n} - EY(t)) \right|^2 \right]^{1/2} &\leq \left[ E \left| \sqrt{\frac{b_n}{n}} m_n \sum_{k=1}^n \int_{A_{l,m_n}} (Y_k(u) - EY_k(u)) du \right|^2 \right]^{1/2} + \\ &\quad + \left| \sqrt{\frac{b_n}{n}} m_n \sum_{k=1}^n \int_{A_{l,m_n}} (EY_k(u) - EY(t)) du \right|. \end{aligned}$$

First we show the convergence to zero of the last term.

Since  $Y$  is periodically correlated, we get

$$\left| \sqrt{\frac{b_n}{n}} m_n \sum_{k=1}^n \int_{A_{l,m_n}} (EY_1(u) - EY(t)) du \right| = \left| (b_n n)^{1/2} m_n \int_{A_{l,m_n}} (EY(u) - EY(t)) du \right|.$$

Due to assumption **A6** we have

$$\left| (b_n n)^{1/2} m_n \int_{A_{l,m_n}} (EY(u) - EY(t)) du \right| \leq C \sqrt{nb_n b_n},$$

where  $C$  is a positive constant independent of  $n$ . From assumption **A7** we get the required convergence to zero.

Now we consider the following expression

$$E \left| \sqrt{\frac{b_n}{n}} m_n \sum_{k=1}^n \int_{A_{l,m_n}} (Y_k(u) - EY_k(u)) du \right|^2. \tag{3.9}$$

Notice that

$$\begin{aligned} &E \left( \sqrt{\frac{b_n}{n}} m_n \sum_{k=1}^n \int_{A_{l,m_n}} (Y_k(u) - EY_k(u)) du \right)^2 = \\ &= \frac{b_n m_n^2}{n} \sum_{k=1}^n \int_{A_{l,m_n}} \int_{A_{l,m_n}} E (Y_k(u) - EY_k(u)) (Y_k(v) - EY_k(v)) dv du + \\ &+ \frac{b_n m_n^2}{n} \sum_{k=1}^n \sum_{k'=1, k' \neq k}^n \int_{A_{l,m_n}} \int_{A_{l,m_n}} E (Y_k(u) - EY_k(u)) (Y_{k'}(v) - EY_{k'}(v)) dv du. \end{aligned}$$

Due to **A1** the right-hand side of the last equation can be rewritten in the following way

$$b_n m_n^2 \int_{A_{l,m_n}} \int_{A_{l,m_n}} \mathbb{E} (Y(u) - \mathbb{E}Y(u)) (Y(v) - \mathbb{E}Y(v)) \, dudv + \\ + \frac{b_n m_n^2}{n} \sum_{k=1}^n \sum_{k'=1, k' \neq k}^n \int_{A_{l,m_n}} \int_{A_{l,m_n}} \text{Cov} (Y_k(u), Y_{k'}(v)) \, dudv.$$

By **A2** we get that the first term is bounded from above by  $D_1 b_n$ , where  $D_1$  is a positive constant independent of  $n$ . We denote the second term by  $II$ . Since  $Y$  is  $\alpha$ -mixing and has bounded moments by Corollary A.2. from [8] we get

$$|II| \leq \frac{b_n m_n^2}{n} \sum_{k=1}^n \sum_{k'=1, k' \neq k}^n \int_{A_{l,m_n}} \int_{A_{l,m_n}} C_1 \alpha^{1-2/p^*} (|k - k'|) \, dudv,$$

where  $p^*$  is the order of a suitably chosen moment of  $Y$  (here  $p^* = 5$ ) and  $C_1$  is a positive constant independent of  $n$ . By **A3** we get  $II \leq D_2 b_n$ , where  $D_2$  is a positive constant independent of  $n$ . This fact ends the proof of the convergence of (3.8) to zero.

Now observe that

$$\mathbb{E} \left| (nb_n)^{1/2} (L_n(t) - \hat{\eta}_n(t)) \right| \leq \\ \leq \sum_{l=1}^{m_n} \mathbb{E} \left| (nb_n)^{1/2} \frac{W_{l,n}(U_{l,n} - \mathbb{E}Y(t))}{U_{l,n} \mathbb{E}Y(t)} \right| \frac{1}{b_n} \int_{A_{l,m_n}} K \left( \frac{t-s}{b_n} \right) \, ds \leq \\ \leq \sum_{l=1}^{m_n} \frac{1}{\mathbb{E}Y(t)} \frac{1}{b_n} \int_{A_{l,m_n}} K \left( \frac{t-s}{b_n} \right) \, ds \sqrt{\mathbb{E} \left| (nb_n)^{1/2} (U_{l,n} - \mathbb{E}Y(t)) \right|^2 \mathbb{E} \left( \frac{W_{l,n}}{U_{l,n}} \right)^2}.$$

By **A1** we get that  $U_{l,n}^2 \geq \delta^2$  and

$$\mathbb{E} \left( \frac{W_{l,n}}{U_{l,n}} \right)^2 \leq \frac{\mathbb{E}W_{l,n}^2}{\delta^2}.$$

Additionally, because for  $w \neq v$

$$\mathbb{E} (X_w(A_{l,m_n}) X_v(A_{l,m_n})) \leq F_0 m_n^{-3/2} \quad (3.10)$$

we have

$$\mathbb{E}W_{l,n}^2 = \frac{m_n^2}{n^2} \sum_{k=1}^n \sum_{k'=1, k' \neq k}^n \mathbb{E} (X_k(A_{l,m_n}) X_{k'}(A_{l,m_n})) + \frac{m_n^2}{n^2} \sum_{k=1}^n \mathbb{E} (X_k^2(A_{l,m_n})) \leq \\ \leq F_1 \sqrt{m_n} + F_2 \frac{m_n}{n},$$

where  $F_1, F_2$  are positive constants independent of  $n$ .

Finally,

$$E \left| (nb_n)^{1/2} (U_{l,n} - EY(t)) \right|^2 E \left( \frac{W_{l,n}}{U_{l,n}} \right)^2 \leq F_3 \sqrt{m_n} nb_n^3, \tag{3.11}$$

which together with (3.5), **A4** and **A7** completes the proof of the lemma. □

**Lemma 3.4.** *Under **A1–A7***

$$\frac{b_n m_n^2}{n} \sum_{k=1}^n \sum_{l=1}^{m_n} \frac{1}{(EY(t))^2} M_k(A_{l,m_n}) \left( \frac{1}{b_n} \int_{A_{l,m_n}} K \left( \frac{t-s}{b_n} \right) ds \right)^2 \xrightarrow{p} 0.$$

*Proof.* We define the martingale array  $\{S_{ni}, 1 \leq i \leq n\}$ , where

$$S_{ni} = \frac{b_n m_n^2}{n} \sum_{k=1}^i \sum_{l=1}^{m_n} \frac{1}{(EY(t))^2} M_k(A_{l,m_n}) \left( \frac{1}{b_n} \int_{A_{l,m_n}} K \left( \frac{t-s}{b_n} \right) ds \right)^2.$$

Due to Theorem 3.2 from [8] in order to obtain desired convergence it is sufficient to verify the following conditions for  $H_{ni} = S_{ni} - S_{n,i-1}$ :

- (i)  $\max_i |H_{ni}| \xrightarrow{p} 0$ ,
- (ii)  $\sum_i H_{ni}^2 \xrightarrow{p} 0$ ,
- (iii)  $E(\max_i H_{ni}^2)$  is bounded.

To get (i), (ii) and (iii) it is enough to show that  $\sum_{i=1}^n E(H_{ni}^2) \rightarrow 0$ .

By **A1**, **A5**, **A7** and the fact that the increments of a martingale are uncorrelated we get

$$\begin{aligned} \sum_i E(H_{ni}^2) &= \frac{b_n^2 m_n^4}{n^2} \sum_{k=1}^n \sum_{l=1}^{m_n} \frac{1}{(EY(t))^4} EM_k^2(A_{l,m_n}) \left( \frac{1}{b_n} \int_{A_{l,m_n}} K \left( \frac{t-s}{b_n} \right) ds \right)^4 = \\ &= \frac{b_n^2 m_n^4}{n^2} \sum_{k=1}^n \sum_{l=1}^{m_n} \frac{1}{(EY(t))^4} \int_{A_{l,m_n}} \lambda_0(u) EY_k(u) du \left( \frac{1}{b_n} \int_{A_{l,m_n}} K \left( \frac{t-s}{b_n} \right) ds \right)^4 \leq \\ &\leq \frac{b_n^2 m_n^4}{n^2} n C_2 \frac{1}{m_n} \frac{1}{(b_n m_n)^3} = \frac{C_2}{nb_n}, \end{aligned}$$

where  $C_2$  is a positive constant independent of  $n$ . This ends the proof. □

Now we prove Theorem 3.1.

*Proof.* Since  $L_n(t)$  and  $\hat{\eta}_n(t)$  are asymptotically equivalent (Lemma 3.3), we show that the limit distribution of  $Z_n(t) = (nb_n)^{1/2} (L_n(t) - \lambda_0(t))$  is asymptotically normal with mean zero and variance given by (3.2).



Using the Doob-Meyer decomposition for our counting process  $X$ , we can rewrite  $Z_n(t)$  as follows

$$\begin{aligned} Z_n(t) &= \sqrt{nb_n} \left( \frac{1}{\text{EY}(t)} \sum_{l=1}^{m_n} W_{l,n} \left( \frac{1}{b_n} \int_{A_{l,m_n}} K \left( \frac{t-s}{b_n} \right) ds \right) - \lambda_0(t) \right) = \\ &= \sqrt{nb_n} \frac{m_n}{n} \frac{1}{\text{EY}(t)} \sum_{k=1}^n \sum_{l=1}^{m_n} M_k(A_{l,m_n}) \left( \frac{1}{b_n} \int_{A_{l,m_n}} K \left( \frac{t-s}{b_n} \right) ds \right) + \\ &\quad + \sqrt{nb_n} \frac{m_n}{n} \frac{1}{\text{EY}(t)} \sum_{k=1}^n \sum_{l=1}^{m_n} \Lambda_k(A_{l,m_n}) \left( \frac{1}{b_n} \int_{A_{l,m_n}} K \left( \frac{t-s}{b_n} \right) ds \right) - \sqrt{nb_n} \lambda_0(t). \end{aligned}$$

Denote

$$I = \sqrt{nb_n} \frac{m_n}{n} \frac{1}{\text{EY}(t)} \sum_{k=1}^n \sum_{l=1}^{m_n} \Lambda_k(A_{l,m_n}) \left( \frac{1}{b_n} \int_{A_{l,m_n}} K \left( \frac{t-s}{b_n} \right) ds \right) - \sqrt{nb_n} \lambda_0(t).$$

First we show that  $I$  converges to zero in probability.

The compensator  $\Lambda_k$  of the counting process  $X_k$  on the interval  $A_{l,m_n}$  is of the form  $\Lambda_k(A_{l,m_n}) = \int_{A_{l,m_n}} \lambda_0(s) Y_k(s) ds$  (for more details see [2], pp. 72–77). Notice that

$$\begin{aligned} I &= \sqrt{\frac{b_n}{n}} \frac{m_n}{\text{EY}(t)} \sum_{k=1}^n \sum_{l=1}^{m_n} \int_{A_{l,m_n}} \lambda_0(s) (Y_k(s) - \text{E}Y_k(s)) ds \left( \frac{1}{b_n} \int_{A_{l,m_n}} K \left( \frac{t-s}{b_n} \right) ds \right) + \\ &\quad + \sqrt{\frac{b_n}{n}} \frac{m_n}{\text{EY}(t)} \sum_{k=1}^n \sum_{l=1}^{m_n} \int_{A_{l,m_n}} \lambda_0(s) \text{E}Y_k(s) ds \left( \frac{1}{b_n} \int_{A_{l,m_n}} K \left( \frac{t-s}{b_n} \right) ds \right) - \sqrt{nb_n} \lambda_0(t). \end{aligned}$$

The hazard function  $\lambda_0(t)$  and the inverse of the mean function  $\text{EY}(t)$  are bounded so we get

$$\begin{aligned} |I| &\leq C_1 \sum_{l=1}^{m_n} \left| \sqrt{\frac{b_n}{n}} m_n \sum_{k=1}^n \int_{A_{l,m_n}} (Y_k(s) - \text{E}Y_k(s)) ds \right| \left( \frac{1}{b_n} \int_{A_{l,m_n}} K \left( \frac{t-s}{b_n} \right) ds \right) + \\ &\quad + \left| \sqrt{\frac{b_n}{n}} \frac{m_n}{\text{EY}(t)} \sum_{k=1}^n \sum_{l=1}^{m_n} \int_{A_{l,m_n}} \lambda_0(s) \text{E}Y_k(s) ds \left( \frac{1}{b_n} \int_{A_{l,m_n}} K \left( \frac{t-s}{b_n} \right) ds \right) - \sqrt{nb_n} \lambda_0(t) \right|, \end{aligned}$$

where  $C_1$  is a positive constant independent of  $n$ .

In the proof of Lemma 3.3 it was shown that

$$\mathbb{E} \left( \sqrt{b_n/nm_n} \sum_{k=1}^n \int_{A_{l,m_n}} (Y_k(s) - \mathbb{E}Y_k(s)) ds \right)^2 \leq C_2 b_n,$$

where  $C_2$  is a positive constant independent of  $n$ . As a consequence we have

$$\mathbb{E} \left| \sqrt{b_n/nm_n} \sum_{k=1}^n \int_{A_{l,m_n}} (Y_k(s) - \mathbb{E}Y_k(s)) ds \right| \leq \sqrt{C_2 b_n}.$$

Given the equality (3.5) the first term on the right-hand side of the considered sum tends to zero in probability.

Now we consider the second term. Under assumption **A1** and (3.5) we get, for large  $n$

$$\begin{aligned} III &= \left| \frac{\sqrt{b_n n m_n}}{\mathbb{E}Y(t)} \sum_{l=1}^{m_n} \int_{A_{l,m_n}} \lambda_0(s) \mathbb{E}Y(s) ds \left( \frac{1}{b_n} \int_{A_{l,m_n}} K \left( \frac{t-s}{b_n} \right) ds \right) - \sqrt{nb_n} \lambda_0(t) \right| \leq \\ &\leq \frac{\sqrt{nb_n}}{\mathbb{E}Y(t)} \sum_{l=1}^{m_n} m_n \int_{A_{l,m_n}} |\lambda_0(s) \mathbb{E}Y(s) - \lambda_0(t) \mathbb{E}Y(t)| ds \left( \frac{1}{b_n} \int_{A_{l,m_n}} K \left( \frac{t-s}{b_n} \right) ds \right). \end{aligned}$$

Moreover, by **A6** we have

$$|\lambda_0(s) \mathbb{E}Y(s) - \lambda_0(t) \mathbb{E}Y(t)| \leq D b_n, \quad (3.12)$$

where  $D$  is a positive constant independent of  $n$ .

Due to (3.12) and (3.5)

$$III \leq \frac{D}{\mathbb{E}Y(t)} \sqrt{nb_n} b_n.$$

This fact together with **A2** and **A7** gives us the required convergence of  $I$  to zero in probability.

To show asymptotic normality of  $Z_n(t)$  we apply the martingale version of the central limit theorem ([8], Theorem 3.2 p.58) for the martingale array  $\{S_{ni}, 1 \leq i \leq n\}$ , where

$$S_{ni} = \sqrt{\frac{b_n}{n}} m_n \sum_{k=1}^i \sum_{l=1}^{m_n} \frac{1}{\mathbb{E}Y(t)} M_k(A_{l,m_n}) \left( \frac{1}{b_n} \int_{A_{l,m_n}} K \left( \frac{t-s}{b_n} \right) ds \right).$$

To check if the assumptions of this theorem are fulfilled, we consider the differences

$$A_{ni} = S_{ni} - S_{n(i-1)} = \sqrt{\frac{b_n}{n}} m_n \sum_{l=1}^{m_n} \frac{1}{EY(t)} M_i(A_{l,m_n}) \left( \frac{1}{b_n} \int_{A_{l,m_n}} K\left(\frac{t-s}{b_n}\right) ds \right).$$

We need to show that the following three conditions are fulfilled:

- (i)  $\max_i |A_{ni}| \xrightarrow{P} 0$ ,
- (ii)  $\sum_i A_{ni}^2 \xrightarrow{P} \sigma^2(t)$ ,
- (iii)  $E(\max_i A_{ni}^2)$  is bounded.

First we consider (ii). Initially we show that  $\sum_i EA_{ni}^2 \rightarrow \sigma^2(t)$  and subsequently that  $\sum_i (A_{ni}^2 - EA_{ni}^2)$  tends to zero in probability.

The expectations  $E(M_k(A_{l,m_n})M_k(A_{l',m_n}))$  are equal to zero for  $l \neq l'$ , so after simple calculations we get

$$\sum_i EA_{ni}^2 = \frac{b_n m_n^2}{n} \sum_{k=1}^n \sum_{l=1}^{m_n} \frac{1}{(EY(t))^2} EM_k^2(A_{l,m_n}) \left( \frac{1}{b_n} \int_{A_{l,m_n}} K\left(\frac{t-s}{b_n}\right) ds \right)^2.$$

Since the compensator of the submartingale  $M_k^2(t)$  is  $\Lambda_k(t) = \int_0^t \lambda_0(s) Y_k(s) ds$  (see [2], p. 74), we have

$$\sum_i EA_{ni}^2 = \frac{b_n m_n^2}{n} \sum_{k=1}^n \sum_{l=1}^{m_n} \frac{1}{(EY(t))^2} \int_{A_{l,m_n}} \lambda_0(s) EY_k(s) ds \left( \frac{1}{b_n} \int_{A_{l,m_n}} K\left(\frac{t-s}{b_n}\right) ds \right)^2,$$

which under **A1** is equal to

$$m_n^2 b_n \sum_{l=1}^{m_n} \frac{1}{(EY(t))^2} \int_{A_{l,m_n}} \lambda_0(s) EY(s) ds \left( \frac{1}{b_n} \int_{A_{l,m_n}} K\left(\frac{t-s}{b_n}\right) ds \right)^2.$$

By (3.12) and (3.6) we have

$$\begin{aligned}
& m_n b_n \left| \sum_{l=1}^{m_n} \frac{1}{(\mathbb{E}Y(t))^2} \left( m_n \int_{A_l, m_n} \lambda_0(s) \mathbb{E}Y(s) ds - \lambda_0(t) \mathbb{E}Y(t) \right) \right. \\
& \quad \left. \cdot \left( \frac{1}{b_n} \int_{A_l, m_n} K \left( \frac{t-s}{b_n} \right) ds \right)^2 \right| \leq \\
& \leq m_n b_n \sum_{l=1}^{m_n} \frac{1}{(\mathbb{E}Y(t))^2} \left( m_n \int_{A_l, m_n} |\lambda_0(s) \mathbb{E}Y(s) - \lambda_0(t) \mathbb{E}Y(t)| ds \right) \\
& \quad \cdot \left( \frac{1}{b_n} \int_{A_l, m_n} K \left( \frac{t-s}{b_n} \right) ds \right)^2 \leq \\
& \leq b_n D_0 \frac{m_n b_n}{(\mathbb{E}Y(t))^2} \frac{1}{m_n b_n} \leq D_1 b_n,
\end{aligned}$$

where  $D_0$  and  $D_1$  are positive constants independent of  $n$ . To get the convergence  $\sum_i \mathbb{E}A_{ni}^2 \rightarrow \sigma^2(t)$  it is enough to show

$$\left| m_n b_n \frac{\lambda_0(t)}{\mathbb{E}Y(t)} \sum_{l=1}^{m_n} \left( \frac{1}{b_n} \int_{A_l, m_n} K \left( \frac{t-s}{b_n} \right) ds \right)^2 - \sigma^2(t) \right| \rightarrow 0$$

or equivalently

$$\left| m_n b_n \sum_{l=1}^{m_n} \left( \frac{1}{b_n} \int_{A_l, m_n} K \left( \frac{t-s}{b_n} \right) ds \right)^2 - \int_{-1}^1 K^2(u) du \right| \rightarrow 0. \quad (3.13)$$

Notice that for large  $n$

$$\int_{-1}^1 K^2(u) du = \sum_{l=1}^{m_n} \frac{1}{b_n} \int_{A_l, m_n} K^2((t-w)/b_n) dw$$

and (3.13) is less or equal to

$$\sum_{l=1}^{m_n} \left| m_n b_n \left( \frac{1}{b_n} \int_{A_l, m_n} K \left( \frac{t-s}{b_n} \right) ds \right)^2 - \frac{1}{b_n} \int_{A_l, m_n} K^2 \left( \frac{t-w}{b_n} \right) dw \right|.$$

From the mean value theorem for integration the above expression can be rewritten as

$$\sum_{l=1}^{m_n} \left| m_n b_n \left( \frac{1}{m_n b_n} K \left( \frac{t - \zeta_l}{b_n} \right) \right)^2 - \frac{1}{m_n b_n} K^2 \left( \frac{t - \xi_l}{b_n} \right) \right|, \quad (3.14)$$

where  $\zeta_l$  and  $\xi_l$  are the intermediate points that belong to  $A_{l,m_n}$ .

Since the function  $K$  fulfills the Lipschitz condition, the number of nonzero summands in the above sum is  $2b_n m_n$  and  $|\xi_l - \zeta_l| \leq 1/m_n$ , (3.14) can be bounded by

$$C_3 \frac{1}{m_n b_n} \sum_{l=1}^{m_n} \left| \frac{\xi_l - \zeta_l}{b_n} \right| \leq \frac{2C_3}{m_n b_n},$$

where  $C_3$  is a positive constant independent of  $n$ . This means that  $\sum_i \mathbb{E} A_{ni}^2 \rightarrow \sigma^2(t)$ .

To show that (ii) is fulfilled we need to show additionally that  $\sum_{i=1}^n (A_{ni}^2 - \mathbb{E}(A_{ni}^2))$  tends to zero in probability.

We can rewrite the sum in question as follows

$$\begin{aligned} & \sum_{k=1}^n \left( \sqrt{\frac{b_n}{n}} m_n \sum_{l=1}^{m_n} \frac{1}{\mathbb{E}Y(t)} M_k(A_{l,m_n}) \left( \frac{1}{b_n} \int_{A_{l,m_n}} K \left( \frac{t-s}{b_n} \right) ds \right) \right)^2 - \\ & - \frac{b_n m_n^2}{n} \sum_{k=1}^n \sum_{l=1}^{m_n} \frac{1}{(\mathbb{E}Y(t))^2} \mathbb{E} M_k^2(A_{l,m_n}) \left( \frac{1}{b_n} \int_{A_{l,m_n}} K \left( \frac{t-s}{b_n} \right) ds \right)^2 = \\ & = \sum_{k=1}^n \frac{b_n}{n} m_n^2 \sum_{l=1}^{m_n} \frac{1}{(\mathbb{E}Y(t))^2} (M_k^2(A_{l,m_n}) - \mathbb{E} M_k^2(A_{l,m_n})) \left( \frac{1}{b_n} \int_{A_{l,m_n}} K \left( \frac{t-s}{b_n} \right) ds \right)^2 + \\ & + \sum_{k=1}^n \frac{b_n}{n} m_n^2 \sum_{l=1}^{m_n} \sum_{l'=1, l' \neq l}^{m_n} \frac{1}{(\mathbb{E}Y(t))^2} M_k(A_{l,m_n}) M_k(A_{l',m_n}) \cdot \\ & \cdot \left( \frac{1}{b_n} \int_{A_{l,m_n}} K \left( \frac{t-s}{b_n} \right) ds \right) \left( \frac{1}{b_n} \int_{A_{l',m_n}} K \left( \frac{t-w}{b_n} \right) dw \right). \end{aligned}$$

We denote the summands of the right-hand side by  $IV$  and  $V$ , respectively.

First we show the convergence of  $V$  to zero in probability.

Using the Doob-Meyer decomposition we get

$$\begin{aligned} M_k(A_{l,m_n}) M_k(A_{l',m_n}) &= X_k(A_{l,m_n}) X_k(A_{l',m_n}) - \Lambda_k(A_{l,m_n}) M_k(A_{l',m_n}) - \\ &- \Lambda_k(A_{l',m_n}) \Lambda_k(A_{l,m_n}) - \Lambda_k(A_{l',m_n}) M_k(A_{l,m_n}). \end{aligned}$$

Moreover,

$$\begin{aligned} \mathbb{E}(X_k(A_{l,m_n})X_k(A_{l',m_n})) &\leq F_1 m_n^{-3/2}, \\ \mathbb{E}(\Lambda_k(A_{l',m_n})\Lambda_k(A_{l,m_n})) &\leq F_2 m_n^{-2}, \\ \mathbb{E}|\Lambda_k(A_{l,m_n})M_k(A_{l',m_n})| &\leq F_3 m_n^{-3/2}, \end{aligned}$$

where the positive constants  $F_1, F_2, F_3$  are independent of  $n$ .

From (3.5) we get

$$\begin{aligned} \mathbb{E}|V| &\leq \sum_{k=1}^n \frac{b_n}{n} m_n^2 \sum_{l=1}^{m_n} \sum_{l'=1, l' \neq l}^{m_n} \frac{F_4}{m_n^{3/2}} \left( \frac{1}{b_n} \int_{A_{l,m_n}} K\left(\frac{t-s}{b_n}\right) ds \right) \\ &\quad \cdot \left( \frac{1}{b_n} \int_{A_{l',m_n}} K\left(\frac{t-w}{b_n}\right) dw \right) \leq F_5 b_n m_n^{1/2}, \end{aligned}$$

where  $F_4, F_5$  are positive constants independent of  $n$ .

This fact together with **A7** gives the required convergence of  $V$  to zero in probability.

To get the convergence of  $IV$  notice that

$$\begin{aligned} M_k^2(A_{l,m_n}) - \mathbb{E}M_k^2(A_{l,m_n}) &= X_k^2(A_{l,m_n}) - X_k(A_{l,m_n}) - \\ &\quad - 2M_k(A_{l,m_n})\Lambda_k(A_{l,m_n}) - \Lambda_k^2(A_{l,m_n}) + M_k(A_{l,m_n}) + \Lambda_k(A_{l,m_n}) - \mathbb{E}\Lambda_k(A_{l,m_n}). \end{aligned}$$

Bearing in mind the above calculations and the fact that

$$\mathbb{E}|X_i^2(A_{l,m_n}) - X_i(A_{l,m_n})| \leq F_6 m_n^{-3/2}, \tag{3.15}$$

we have

$$\mathbb{E}|X_k^2(A_{l,m_n}) - X_k(A_{l,m_n}) - 2M_k(A_{l,m_n})\Lambda_k(A_{l,m_n}) - \Lambda_k^2(A_{l,m_n})| \leq F_7 m_n^{-3/2},$$

where  $F_6, F_7$  are positive constants independent of  $n$ .

Under Lemma 3.4 we only need to get the convergence to zero in probability of

$$\begin{aligned} &\sum_{k=1}^n \frac{b_n m_n^2}{n} \sum_{l=1}^{m_n} \frac{1}{(\mathbb{E}Y(t))^2} (\Lambda_k(A_{l,m_n}) - \mathbb{E}\Lambda_k(A_{l,m_n})) \left( \frac{1}{b_n} \int_{A_{l,m_n}} K\left(\frac{t-s}{b_n}\right) ds \right)^2 = \\ &= \sum_{k=1}^n \frac{b_n m_n^2}{n} \sum_{l=1}^{m_n} \frac{1}{(\mathbb{E}Y(t))^2} \int_{A_{l,m_n}} \lambda_0(u)(Y_k(u) - \mathbb{E}Y_k(u)) du \left( \frac{1}{b_n} \int_{A_{l,m_n}} K\left(\frac{t-s}{b_n}\right) ds \right)^2, \end{aligned}$$

which will be denoted by  $VI$ .

In the proof of Lemma 3.3 we get that

$$\mathbb{E} \left| \sqrt{b_n/nm_n} \sum_{k=1}^n \int_{A_{l,m_n}} (Y_k(s) - \mathbb{E}Y_k(s)) ds \right| \leq F_8 \sqrt{b_n},$$

where  $F_8$  is a positive constant independent of  $n$ . Under **A2**, **A5** and (3.6) we have

$$\mathbb{E}|VI| \leq F_9 \sqrt{\frac{b_n}{n}} m_n \sqrt{b_n} \frac{1}{b_n m_n} = \frac{F_9}{\sqrt{n}},$$

where  $F_9$  is a positive constant independent of  $n$ . This ends the proof of (ii).

The condition (iii) may be obtained by noticing that under **A2**, **A5**, **A6** and (3.6) we have

$$\begin{aligned} \mathbb{E} \left( \max_i A_{ni}^2 \right) &\leq \sum_{i=1}^n \mathbb{E}(A_{ni}^2) = \\ &= m_n^2 b_n \sum_{l=1}^{m_n} \frac{1}{(\mathbb{E}Y(t))^2} \int_{A_{l,m_n}} \lambda_0(s) \mathbb{E}Y(s) ds \left( \frac{1}{b_n} \int_{A_{l,m_n}} K \left( \frac{t-s}{b_n} \right) ds \right)^2 \leq F_{10}, \end{aligned}$$

where  $F_{10}$  is a positive constant independent of  $n$ .

To get (i) notice that

$$\begin{aligned} P \left( \max_i |A_{ni}| > \varepsilon \right) &\leq \\ &\leq \sum_{i=1}^n \sum_{l=1}^{m_n} P \left( \sqrt{\frac{b_n}{n}} m_n \frac{1}{\mathbb{E}Y(t)} |M_i(A_{l,m_n})| \left( \frac{1}{b_n} \int_{A_{l,m_n}} K \left( \frac{t-s}{b_n} \right) ds \right) > \varepsilon \right) \leq \\ &\leq \sum_{i=1}^n \sum_{l=1}^{m_n} P \left( X_i(A_{l,m_n}) > \frac{\varepsilon \sqrt{nb_n}}{C_1} \right) + \sum_{i=1}^n \sum_{l=1}^{m_n} P \left( \Lambda_i(A_{l,m_n}) > \frac{\varepsilon \sqrt{nb_n}}{C_1} \right), \end{aligned}$$

where  $C_1$  is a positive constant independent of  $n$  and the summands are nonzero for only  $2m_n b_n$  values of  $l$ .

Since the stochastic intensity  $\lambda(t)$  of the counting process  $X$  is bounded by a constant ( $\lambda(t) \leq C$ ), the process  $X$  is dominated by the Poisson process with intensity  $C$  (for more details see [9] and [15]). As a consequence we get

$$P \left( X_i(A_{l,m_n}) > \frac{\varepsilon \sqrt{nb_n}}{C_1} \right) \leq \frac{C^{w+1}}{m_n^{w+1}} \frac{1}{1 - C/m_n}$$

and

$$\sum_{i=1}^n \sum_{l=1}^{m_n} P \left( X_i(A_{l,m_n}) > \frac{\varepsilon \sqrt{nb_n}}{C_1} \right) \leq 2nm_n b_n \frac{C^{w+1}}{m_n^{w+1}} \frac{1}{1 - C/m_n},$$

where  $w = \lfloor \frac{\varepsilon \sqrt{nb_n}}{C_1} \rfloor$ .

The right-hand side of the last inequality tends to zero as  $n \rightarrow \infty$ .  
 Moreover, under **A1**, **A2**, **A5** and **A6** we get

$$\sum_{i=1}^n \sum_{l=1}^{m_n} P \left( \Lambda_i(A_{l,m_n}) > \frac{\varepsilon \sqrt{nb_n}}{C_1} \right) \leq \sum_{i=1}^n \sum_{l=1}^{m_n} \frac{E \left( \int_{A_{l,m_n}} \lambda_0(s) Y_i(s) ds \right)^2}{\frac{\varepsilon^2 nb_n}{C_1^2}} \leq \frac{F_{11}}{m_n},$$

where  $F_{11}$  is a positive constant independent of  $n$ .

This means that  $P(\max_i |A_{ni}| > \varepsilon)$  tends to zero as  $n \rightarrow \infty$ , which ends the proof of (i) and of the theorem. □

Now we present the multidimensional version of Theorem 3.1, which is the crucial result to construct simultaneous confidence intervals for the periodic hazard function.

**Theorem 3.5.** *Under **A1–A7** for a finite set of the time moments  $\{t_1, \dots, t_p\}$*

$$(V_n(t_1), \dots, V_n(t_p))^{Tr} \xrightarrow{d} N_p(0, \Sigma),$$

where  $V_n(t_i) = \sqrt{nb_n} (\hat{\eta}_n(t_i) - \lambda_0(t_i))$ ,  $i = 1, \dots, p$  and  $\Sigma$  is the diagonal matrix of the size  $p \times p$ , and the  $i$ -th diagonal element is of the form

$$\sigma^2(t_i) = \frac{\lambda_0(t_i)}{EY(t_i)} \int_{-1}^1 K^2(u) du.$$

Notice that the covariance matrix is diagonal. This means that  $V_n(t_i)$  and  $V_n(t_j)$  ( $i \neq j$ ) are asymptotically independent and normal.

*Proof.* To get the claim we need to use the Cramer-Wold device. The linear combination  $s_1 V_n(t_1) + \dots + s_p V_n(t_p)$  has the asymptotic distribution  $N(0, \sum_{i=1}^p s_i^2 \sigma^2(t_i))$ . The proof of this fact is analogous to the proof of Theorem 3.1. The martingale array is now of the form  $\{s_1 S_{ni}(t_1) + \dots + s_p S_{ni}(t_p) : 1 \leq i \leq n\}$ , where

$$S_{ni}(t_j) = \frac{b_n m_n^2}{n} \sum_{k=1}^i \sum_{l=1}^{m_n} \frac{1}{(EY(t_j))^2} M_k(A_{l,m_n}) \left( \frac{1}{b_n} \int_{A_{l,m_n}} K \left( \frac{t_j - s}{b_n} \right) ds \right)^2.$$

We omit the rest of the technical details. □

In the next section we present the key results of this paper. We construct the bootstrap version of the smoothed estimator of the periodic hazard function. We show the consistency of the considered bootstrap scheme in the one and the multidimensional case.



## 4. CONSISTENCY OF BOOTSTRAP

To construct the bootstrap version of the estimator  $\hat{\eta}_n(t)$  we use the algorithm proposed by Dudek and Leśkow in [6]. The phase-consistent resampling scheme (*PCRS*) is designed for the counting processes which intensity function has some periodicity properties. It is the modification of the algorithm presented by Braun and Kulperger in [3], which was dedicated to the stationary case. The authors of [6] used *PCRS* to construct the bootstrap version of the periodic hazard function estimator and as a consequence the bootstrap simultaneous confidence intervals for this function. They show that these bands perform very well which means that their actual coverage probabilities are very close to nominal ones independently of the considered hazard function and the scheme of generating the process  $Y$ .

The bootstrap version of the estimator  $\hat{\eta}_n(t)$  is defined as follows

$$\hat{\eta}_n^*(t) = \frac{1}{Pb_n} \int_0^P K\left(\frac{t-s}{Pb_n}\right) \hat{\lambda}_n^*(s) ds, \quad (4.1)$$

where  $\hat{\lambda}_n^*(s)$  is the bootstrap version of the estimator  $\hat{\lambda}_n(s)$  (for more details see [6])

$$\hat{\lambda}_n^*(s) = \sum_{l=1}^{m_n} \frac{\sum_{k=1}^n X_k^*(A_{l,m_n})}{\sum_{k=1}^n \int_{A_{l,m_n}} Y_k(u) du} 1_{A_{l,m_n}}(s), \quad s \in [0, P]. \quad (4.2)$$

In the theorem below we establish the consistency of the *PCRS* scheme, which means that we show that the percentiles of the bootstrap distribution are uniformly close to the asymptotic ones. This result is the key to obtain the bootstrap simultaneous confidence intervals for the considered hazard function. As in the case of the non-smoothed estimator  $\hat{\lambda}_n(t)$  (see [6]) the percentiles of the asymptotic distribution are quite hard to obtain. Therefore, a need for bootstrap approach appears.

**Theorem 4.1.** *Under A1–A7*

$$\sup_{u \in \mathbf{R}} \left| P^* \left( \sqrt{nb_n} (\hat{\eta}_n^*(t) - \hat{\eta}_n(t)) \leq u \right) - P \left( \sqrt{nb_n} (\hat{\eta}_n(t) - \lambda_0(t)) \leq u \right) \right| = o_P(1).$$

*Proof.* As before, without loss of generality we take  $P = 1$ .

First we present a lemma which turns out to be of great importance for our result.

**Lemma 4.2.** *Let*

$$L_n^*(t) = \frac{1}{EY(t)} \sum_{l=1}^{m_n} W_{l,n}^* \left( \frac{1}{b_n} \int_{A_{l,m_n}} K\left(\frac{t-s}{b_n}\right) ds \right) \quad (4.3)$$

be the bootstrap version of  $L_n(t)$  defined by (3.7).

Then  $L_n^*(t)$  and  $\hat{\eta}_n^*(t)$  are asymptotically equivalent, i.e.  $\sqrt{nb_n} (L_n^*(t) - \hat{\eta}_n^*(t))$  tends to zero in probability.

*Proof.* Notice that

$$\begin{aligned} \sqrt{nb_n} \mathbb{E}^* |L_n^*(t) - \widehat{\eta}_n^*(t)| &= \sqrt{nb_n} \mathbb{E}^* \left| \sum_{l=1}^{m_n} \frac{W_{l,n}^*}{U_{l,n}} \left( \frac{U_{l,n} - \mathbb{E}Y(t)}{\mathbb{E}Y(t)} \right) \frac{1}{b_n} \int_{A_{l,m_n}} K \left( \frac{t-s}{b_n} \right) ds \right| \leq \\ &\leq \sqrt{nb_n} \sum_{l=1}^{m_n} \frac{\mathbb{E}^* (W_{l,n}^*)}{U_{l,n}} \frac{|U_{l,n} - \mathbb{E}Y(t)|}{\mathbb{E}Y(t)} \frac{1}{b_n} \int_{A_{l,m_n}} K \left( \frac{t-s}{b_n} \right) ds. \end{aligned}$$

In the proof of Theorem 4.1 from [6] it is shown that  $\mathbb{E}^* (W_{l,n}^*) = W_{l,n}$ .

Moreover,

$$\sqrt{nb_n} \mathbb{E} \left( \frac{W_{l,n}}{U_{l,n}} |U_{l,n} - \mathbb{E}Y(t)| \right) \leq \sqrt{\mathbb{E} \left( \frac{W_{l,n}}{U_{l,n}} \right)^2 \mathbb{E} \left| \sqrt{nb_n} (U_{l,n} - \mathbb{E}Y(t)) \right|^2},$$

which together with (3.11), (3.5) and **A2** gives the claim of the lemma. □

The key step of this proof is to use the conditional Slutsky’s theorem ([10], Lemma 4.1). Taking under consideration additionally Lemma 4.2 we only need to show that

$$\sup_{u \in \mathbf{R}} \left| P^* \left( \sqrt{nb_n} (L_n^*(t) - \mathbb{E}^*(L_n^*(t))) \leq u \right) - \Phi(u, \sigma^2(t)) \right| = o_P(1),$$

where  $\Phi(u, \sigma^2(t))$  is the value at  $u$  of the cumulative distribution function of the normal distribution with zero mean and variance  $\sigma^2(t)$ .

First we calculate the mean and the variance of  $L_n^*(t)$ .

$$\begin{aligned} \mathbb{E}^* (L_n^*(t)) &= \sum_{l=1}^{m_n} \frac{\mathbb{E}^* (W_{l,n}^*)}{\mathbb{E}Y(t)} \frac{1}{b_n} \int_{A_{l,m_n}} K \left( \frac{t-s}{b_n} \right) ds = \\ &= \sum_{l=1}^{m_n} \frac{W_{l,n}}{\mathbb{E}Y(t)} \frac{1}{b_n} \int_{A_{l,m_n}} K \left( \frac{t-s}{b_n} \right) ds = L_n(t). \end{aligned}$$

Additionally,

$$\text{Var}^* \left( \sqrt{nb_n} L_n^*(t) \right) = \frac{nb_n}{\mathbb{E}^2 Y(t)} \sum_{l=1}^{m_n} \text{Var}^* (W_{l,n}^*) \left( \frac{1}{b_n} \int_{A_{l,m_n}} K \left( \frac{t-s}{b_n} \right) ds \right)^2.$$

In the proof of Theorem 4.1 from [6] it is shown that

$$\mathbb{E} \left| \frac{n}{m_n} \text{Var}^* (W_{l,n}^*) - W_{l,n} \right| \leq F m_n^{-1/2},$$

where  $F$  is a positive constant independent of  $n$ .

This fact together with (3.6) gives us

$$\left| \text{Var}^*(\sqrt{nb_n}L_n^*(t)) - m_nb_n \sum_{l=1}^{m_n} \frac{W_{l,n}}{\text{E}^2Y(t)} \left( \frac{1}{b_n} \int_{A_{l,m_n}} K\left(\frac{t-s}{b_n}\right) ds \right) \right|^2 \xrightarrow{p} 0.$$

Additionally

$$\text{E} |W_{l,n} - \lambda_0(t)\text{E}Y(t)| \leq \text{E} \left| \frac{m_n}{n} \sum_{k=1}^n \Lambda_k(A_{l,m_n}) - \lambda_0(t)\text{E}Y(t) \right| + \text{E} \left| \frac{m_n}{n} \sum_{k=1}^n M_k(A_{l,m_n}) \right|.$$

The summands on the right-hand side are denoted by *VII* and *VIII*, respectively.

Since the increments of the martingale are uncorrelated we have

$$VIII \leq \sqrt{\text{E} \left( \frac{m_n}{n} \sum_{k=1}^n M_k(A_{l,m_n}) \right)^2} = \sqrt{\frac{m_n^2}{n^2} \sum_{k=1}^n \text{E}M_k^2(A_{l,m_n})} \leq \sqrt{F_1 \frac{m_n}{n}},$$

where  $F_1$  is a positive constant independent of  $n$ .

Moreover,

$$\begin{aligned} VII \leq \text{E} & \left| \frac{m_n}{n} \sum_{k=1}^n \int_{A_{l,m_n}} \lambda_0(u) (Y_k(u) - \text{E}Y_k(u)) du \right| + \\ & + \text{E} \left| \frac{m_n}{n} \sum_{k=1}^n \int_{A_{l,m_n}} (\lambda_0(u)\text{E}Y_k(u) - \lambda_0(t)\text{E}Y(t)) du \right|. \end{aligned}$$

Since (3.9) is less or equal to  $F_2b_n$  and taking under consideration (3.12) we get

$$VII \leq \sqrt{\frac{F_2}{n}} + F_3b_n.$$

This means that  $\text{E} |W_{l,n} - \lambda_0(t)\text{E}Y(t)|$  is bounded from above by the expression tending to zero as  $n \rightarrow \infty$ .

Finally,

$$\left| \text{Var}^*(\sqrt{nb_n}L_n^*(t)) - m_nb_n \sum_{l=1}^{m_n} \frac{\lambda_0(t)}{\text{E}Y(t)} \left( \frac{1}{b_n} \int_{A_{l,m_n}} K\left(\frac{t-s}{b_n}\right) ds \right) \right|^2 \xrightarrow{p} 0$$

and by (3.13)

$$\left| \text{Var}^*(\sqrt{nb_n}L_n^*(t)) - \frac{\lambda_0(t)}{\text{E}Y(t)} \int_{-1}^1 K^2(u) du \right| \xrightarrow{p} 0.$$

Using Theorem 3.1 and the Pólya Theorem (see [16] p. 447)

$$\sup_{u \in \mathbf{R}} \left| P \left( \sqrt{nb_n} (L_n(t) - \lambda_0(t)) \leq u \right) - \Phi(u, \sigma^2(t)) \right| \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

We only need to show that

$$\sup_{u \in \mathbf{R}} \left| P^* \left( \sqrt{nb_n} (L_n^*(t) - L_n(t)) \leq u \right) - \Phi(u, \sigma^2(t)) \right| = o_P(1) \quad \text{as } n \rightarrow \infty.$$

We apply a version of Berry–Essen theorem from [16] for independent but non-identically distributed random variables:

$$W_i^* = m_n \sqrt{\frac{b_n}{n}} \sum_{l=1}^{m_n} (X_i^*(A_{l,m_n}) - E^*(X_i^*(A_{l,m_n}))) \left( \frac{1}{b_n} \int_{A_{l,m_n}} K\left(\frac{t-s}{b_n}\right) ds \right) / (EY(t)\hat{\sigma}^2(t)),$$

where  $i = 1, \dots, n$  and  $\hat{\sigma}^2(t) = \frac{\hat{\lambda}_n(t)}{U_{l,n}} \int_{-1}^1 K^2(u) du$  is the estimator of  $\sigma^2(t)$ .

To get the claim of our theorem, it is enough to show the convergence to zero in probability of

$$\begin{aligned} & m_n^3 \left(\frac{b_n}{n}\right)^{3/2} \sum_{i=1}^n E^* \left| \sum_{l=1}^{m_n} (X_i^*(A_{l,m_n}) - E^*(X_i^*(A_{l,m_n}))) \cdot \left( \frac{1}{b_n} \int_{A_{l,m_n}} K\left(\frac{t-s}{b_n}\right) ds \right) \right|^3 = \\ & = m_n^3 \left(\frac{b_n}{n}\right)^{3/2} \sum_{i=1}^n \frac{1}{n} \sum_{w_1=1}^n \dots \frac{1}{n} \sum_{w_{m_n}=1}^n \left| \sum_{l=1}^{m_n} (X_{w_l}(A_{l,m_n}) - \bar{X}(A_{l,m_n})) \cdot \left( \frac{1}{b_n} \int_{A_{l,m_n}} K\left(\frac{t-s}{b_n}\right) ds \right) \right|^3, \end{aligned}$$

where  $\bar{X}(A_{l,m_n}) = \frac{1}{n} \sum_{v=1}^n X_v(A_{l,m_n})$ .

Denote the right-hand side of the above equality by  $IX$ .

Notice that

$$\begin{aligned} & \left( E \left| \sum_{l=1}^{m_n} (X_{w_l}(A_{l,m_n}) - \bar{X}(A_{l,m_n})) \left( \frac{1}{b_n} \int_{A_{l,m_n}} K\left(\frac{t-s}{b_n}\right) ds \right) \right|^3 \right)^{1/3} \leq \\ & \leq \sum_{l=1}^{m_n} \left( E \left| (X_{w_l}(A_{l,m_n}) - \bar{X}(A_{l,m_n})) \left( \frac{1}{b_n} \int_{A_{l,m_n}} K\left(\frac{t-s}{b_n}\right) ds \right) \right|^3 \right)^{1/3}. \end{aligned}$$

Since  $X$  is stochastically dominated by the Poisson process ([9]) we have

$$E(X_w^3(A_{l,m_n})) \leq \frac{C}{m_n} + 3\left(\frac{C}{m_n}\right)^2 + \left(\frac{C}{m_n}\right)^3.$$

Moreover,

$$E(X_w^3(A_{l,m_n})) \leq \frac{D_2}{m_n}, \quad (4.4)$$

where  $D_2$  is a positive constant independent of  $n$ .

As a consequence we get

$$E\left|\left(X_{w_l}(A_{l,m_n}) - \bar{X}(A_{l,m_n})\right) \left(\frac{1}{b_n} \int_{A_{l,m_n}} K\left(\frac{t-s}{b_n}\right) ds\right)\right|^3 \leq \frac{D_3}{m_n(b_n m_n)^3},$$

where  $D_3$  is a positive constant independent of  $n$ .

Additionally,

$$\sum_{l=1}^{m_n} \left( E\left|\left(X_{w_l}(A_{l,m_n}) - \bar{X}(A_{l,m_n})\right) \left(\frac{1}{b_n} \int_{A_{l,m_n}} K\left(\frac{t-s}{b_n}\right) ds\right)\right|^3 \right)^{1/3} \leq \frac{D_4}{\sqrt[3]{m_n}}.$$

where  $D_4$  is a positive constant independent of  $n$ .

Finally,

$$E(IX) \leq D_4^3 \frac{m_n^2 b_n^{3/2}}{\sqrt{n}}, \quad (4.5)$$

which under **A4** and **A7** gives us the convergence of  $IX$  to zero in probability and ends the proof of the theorem.  $\square$

The most important application of the bootstrap technique presented in this paper is the construction of the bootstrap simultaneous confidence intervals. Some confidence regions were proposed for example in [6]. The key result that allows us to prove their consistency is the multidimensional version of Theorem 4.1, which can be found below.

**Theorem 4.3.** *Under **A1–A7***

$$\sup_{u \in \mathbf{R}^p} \left| P^* \left( V^{*(n)}(t) \leq u \right) - P \left( V^{(n)}(t) \leq u \right) \right| = o_P(1),$$

where  $V^{*(n)}(t) = (V_n^*(t_1), \dots, V_n^*(t_p))^{Tr}$  and  $V^{(n)}(t) = (V_n(t_1), \dots, V_n(t_p))^{Tr}$ .

Moreover,

$$V_n(t_i) = \sqrt{nb_n} (\hat{\eta}_n(t_i) - \lambda_0(t_i)), \quad i = 1, \dots, p$$

and

$$V_n^*(t_i) = \sqrt{nb_n} (\hat{\eta}_n^*(t_i) - \hat{\eta}_n(t_i)), \quad i = 1, \dots, p.$$

*Proof.* Since all steps of the proof are quite similar to those presented in [6] (Theorem 4.2) the proof is omitted.  $\square$

In the next section we describe a possible application of our results and some modification of the estimator considered that helps to reduce the edge effects.

## 5. REMARKS AND CONCLUSIONS

While calculating the value of the estimator one wants to have some idea about its accuracy. When a function is estimated the most convenient are the simultaneous confidence bands. Following the authors of [6] and using results presented in Sections 3 and 4 one may construct the consistent simultaneous bootstrap confidence intervals. For the estimator  $\hat{\eta}_n(t)$  (just like in the case of the estimator  $\hat{\lambda}_n(t)$ , see [6]) the simultaneous asymptotic confidence intervals are hard to obtain because the percentiles of the asymptotic distribution are not easy to calculate. That is why the bootstrap confidence intervals are the reasonable alternative.

We constructed all confidence regions proposed by Dudek and Leśkow in [6]. The broad simulation study was made. Since the results are very similar to those presented in [6] we have decided not to present them. The actual coverage probabilities of confidence intervals were very close to nominal ones independently of the shape of the considered periodic hazard function, the method of generating the process  $Y$  and the number of periods that was taken.

This means that the bootstrap simultaneous confidence intervals may be used in the real data applications (for a real data example see [6]) and this paper gives additionally the possibility to calculate them not only in the  $m_n$  time moment like was in the case of  $\hat{\lambda}_n(t)$ .

The estimator  $\hat{\eta}_n(t)$  is meaningful only on the interval  $[Pb_n, P - Pb_n]$ . To improve its behavior near the edges 0 and  $P$  one may use the idea of Leśkow (see [12]) to wrap the interval  $[0, P]$  around the circle and define on it the estimator  $\hat{\eta}_n^T(t)$ . This concept of elimination the edge effect is similar to the one presented by Politis and Romano in [14] for the moving block bootstrap method or the version of tiling (see for example [13]). The idea of Leśkow may be also applied in the case considered in this paper. Then the estimator  $\hat{\eta}_n^T(t)$  is asymptotically normal with the same variance as  $\hat{\eta}_n(t)$  provided that the conditions of Theorem 3.1 are fulfilled.

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