Semyon B. Yakubovich

A DOUBLE INDEX TRANSFORM WITH A PRODUCT OF MACDONALD'S FUNCTIONS REVISITED

Abstract. We prove an inversion theorem for a double index transform, which is associated with the product of Macdonald's functions $K_{i\tau}\left(\sqrt{x^2+y^2}-y\right)K_{i\tau}\left(\sqrt{x^2+y^2}+y\right)$, where $(x,y)\in\mathbb{R}_+\times\mathbb{R}_+$ and $i\tau, \tau\in\mathbb{R}_+$ is a pure imaginary index. The results obtained in the sequel are applied to find particular solutions of integral equations involving the square and the cube of the Macdonald function $K_{i\tau}(t)$ as a kernel.

Keywords: Macdonald function, index transform, Kontorovich-Lebedev transform, double Mellin transform, Plancherel theorem, Parseval equality.

Mathematics Subject Classification: 44A20, 33C10, 35C15.

1. INTRODUCTION AND PRELIMINARY RESULTS

In [8, 9] it was proved that the double integral transform

$$F(\tau)=\lim_{N\to\infty}\int\limits_{1/N}^{N}\int\limits_{1/N}^{N}K_{i\tau}\left(\sqrt{x^2+y^2}-y\right)K_{i\tau}\left(\sqrt{x^2+y^2}+y\right)G(x,y)\frac{dxdy}{x} \tag{1.1}$$

represents a left-inverse operator for the index transform

$$G(x,y)=\left(\frac{2}{\pi}\right)^4\lim_{N\to\infty}\int\limits_0^N\tau\sinh 2\pi\tau K_{i\tau}\left(\sqrt{x^2+y^2}-y\right)K_{i\tau}\left(\sqrt{x^2+y^2}+y\right)F(\tau)d\tau, \tag{1.2}$$

which, in turn, is a bounded operator

$$G: L_2\left(\mathbb{R}_+;\tau\sinh 2\pi\tau d\tau\right)\to L_2\left(\mathbb{R}_+\times\mathbb{R}_+;x^{-1}dxdy\right),$$

where the convergence in (1.1), (1.2) is by norms in Hilbert spaces $L_2(\mathbb{R}_+; \tau \sinh 2\pi\tau d\tau)$, $L_2\left(\mathbb{R}_+ \times \mathbb{R}_+; x^{-1}dxdy\right)$, respectively. Its range does not coincide with $L_2\left(\mathbb{R}_+ \times \mathbb{R}_+; x^{-1}dxdy\right)$, however the isometric Parseval equality holds

$$\int_0^\infty \int_0^\infty |G(x,y)|^2 \frac{dxdy}{x} = \left(\frac{2}{\pi}\right)^4 \int_0^\infty \tau \sinh 2\pi\tau |F(\tau)|^2 d\tau. \tag{1.3}$$

In this paper we will find sufficient conditions for absolute and uniform convergence with respect to $\tau \geq 0$ of the double integral in (1.1), representing a right-inverse operator for (1.2), meaning

$$\begin{aligned} G(x,y) = \frac{2^5}{\pi^4} \lim_{\alpha \to \pi-} \int_0^\infty \tau \sinh \alpha\tau \cosh \pi\tau \times \\ \times K_{i\tau}\left(\sqrt{x^2+y^2} - y\right) K_{i\tau}\left(\sqrt{x^2+y^2} + y\right) F(\tau) d\tau, \end{aligned} \tag{1.4}$$

where the convergence is pointwise. These results will be applied to find solutions in the closed form of Lebedev's type integral equations of the first kind [6, 7]

$$\int_0^\infty S(\tau,t) K_{i\tau}(t) f(t) dt = F(\tau), \quad \tau > 0, \tag{1.5}$$

where $K_\nu(z)$ in (1.1), (1.2), (1.4), (1.5) is the modified Bessel function or Macdonald's function [2, vol. II] and $S(\tau,t)$ is generally a special function of hypergeometric type [2, vol. I]. In particular, we will consider an integral equation involving the cube of the Macdonald function

$$\int_0^\infty K_{i\tau}^3(t) f(t) dt = F(\tau), \quad \tau > 0. \tag{1.6}$$

We note that the equation

$$\int_0^\infty K_{i\tau}(t) f(t) dt = F(\tau), \quad \tau > 0 \tag{1.7}$$

is called the Kontorovich-Lebedev integral equation or transformation [6, 7]. The case of the square of Macdonald's function as the kernel

$$\frac{\pi}{\cosh \pi\tau} \int_0^\infty K_{i\tau}^2(t) f(t) dt = F(\tau), \quad \tau > 0 \tag{1.8}$$

was considered for the first time by Lebedev [3].

The modified function $K_\nu(z)$ satisfies the differential equation

$$z^2\frac{d^2u}{dz^2}+z\frac{du}{dz}-(z^2+\nu^2)u=0, \tag{1.9}$$

for which it is a solution that remains bounded as z tends to infinity on the real line. It has the asymptotic behaviour [2, vol. II]

$$K_\nu(z)=\left(\frac{\pi}{2z}\right)^{1/2}e^{-z}[1+O(1/z)], \qquad z\to\infty, \tag{1.10}$$

and near the origin

$$K_\nu(z)=O\left(z^{-|\mathrm{Re}\nu|}\right), \quad z\to 0, \tag{1.11}$$

$$K_0(z)=O\left(\log\frac{1}{z}\right), \quad z\to 0. \tag{1.12}$$

When $|\tau|\to\infty$ and $x>0$, $\gamma\in\mathbb{R}$ are fixed, the kernel $K_{\gamma+i\tau}(x)$ behaves as (cf. [7, ch. 1])

$$K_{\gamma+i\tau}(x)=\frac{\sqrt{2\pi}e^{\gamma\pi i}}{|\tau|^{\gamma+1/2}}\left(\frac{x}{2}\right)^{\gamma}e^{-\pi|\tau|/2}\sin\left(\tau\left(\log\frac{2|\tau|}{x}-1\right)+\left(\gamma+\frac{1}{2}\right)\frac{\pi}{2}+\frac{x^2}{4|\tau|}\right)\times \\ \times\left(1+O\left(\frac{1}{|\tau|}\right)\right). \tag{1.13}$$

The modified Bessel function can be given by the following integral [2, vol. II]

$$K_\nu(z)=\int_0^\infty e^{-z\cosh u}\cosh \nu u du,\ \mathrm{Re}z>0. \tag{1.14}$$

The product of these functions of different arguments can be represented by the Macdonald formula (cf. [2, vol. II], [7])

$$K_\nu(x)K_\nu(y)=\frac{1}{2}\int_0^\infty e^{-\frac{1}{2}\left(u\frac{x^2+y^2}{xy}+\frac{xy}{u}\right)}K_\nu(u)\frac{du}{u}. \tag{1.15}$$

Letting $\nu=i\tau$ in (1.15) we obtain the following representation for the kernel of transformation (1.1)

$$K_{i\tau}\left(\sqrt{x^2+y^2}-y\right)K_{i\tau}\left(\sqrt{x^2+y^2}+y\right)=\frac{1}{2}\int_0^\infty e^{-\left(2\frac{y^2}{x^2}+1\right)u-\frac{x^2}{2u}}K_{i\tau}(u)\frac{du}{u}. \tag{1.16}$$

In the sequel we are going to employ the following useful relations (see [5], formulas (2.16.51.8), (2.16.53.1), (2.16.6.5))

$$\int_0^\infty \tau \sinh \alpha\tau K_{i\tau}(x) K_{i\tau}(y) d\tau = \tag{1.17}$$
$$= \frac{\pi}{2} xy \sin\alpha \frac{K_1((x^2+y^2+2xy\cos\alpha)^{1/2})}{(x^2+y^2+2xy\cos\alpha)^{1/2}}, x, y > 0, \quad 0 \le \alpha < \pi,$$

$$\int_0^\infty \tau \sinh 2\pi\tau \Gamma(\nu + i\tau)\Gamma(\nu - i\tau) K_{i\tau}(x) K_{i\tau}(y) d\tau = \tag{1.18}$$
$$= \frac{2^\nu \pi^{5/2}}{\Gamma(1/2-\nu)} (xy)^\nu |x-y|^{-\nu} K_\nu(|x-y|), \quad 0 \le \mathrm{Re}\nu < \frac{1}{2},$$

where $\Gamma(z)$ is Euler's gamma-function [2, vol. I],

$$\int_0^\infty t^{\alpha-1} e^t K_{i\tau}(t) dt = \frac{\cosh \pi\tau}{2^\alpha \sqrt{\pi}} \Gamma(\alpha + i\tau)\Gamma(\alpha - i\tau)\Gamma(1/2-\alpha), \tag{1.19}$$

where $0 \le \mathrm{Re}\alpha < \frac{1}{2}$.

We will also appeal to the theory of the one- and two-dimensional Mellin transforms [1], [2, vol. I], [4,6]. In fact, the Mellin transform of one variable is defined by the integral

$$f^{\mathcal{M}}(s) = f^*(s) = \int_0^\infty f(x) x^{s-1} dx, s = \gamma + it, \tag{1.20}$$

for $f \in L_1(\mathbb{R}_+; x^{\gamma-1}dx)$, i.e.

$$\|f\|_1 = \int_0^\infty |f(x)| x^{\gamma-1} dx < +\infty,$$

which maps this space into the space of bounded continuous functions vanishing at infinity. However, if $f \in L_2(\mathbb{R}_+; x^{2\gamma-1}dx)$ with the norm

$$\|f\|_2 = \left(\int_0^\infty |f(x)|^2 x^{2\gamma-1} dx \right)^{1/2} < +\infty,$$

then it forms an isometric isomorphism

$$f^{\mathcal{M}} : L_2(\mathbb{R}_+; x^{2\gamma-1}dx) \leftrightarrow L_2((\gamma - i\infty, \gamma + i\infty); dt),$$

and integral (1.10) converges in the mean square sense. The inverse operator is given by the integral

$$f(x) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} f^{\mathcal{M}}(s) x^{-s} ds, \quad s = \gamma + it, \quad x > 0, \tag{1.21}$$

which is convergent in the mean square sense too. Moreover, the following Parseval equality holds

$$\int_0^\infty |f(x)|^2 x^{2\gamma-1} dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} |f^{\mathcal{M}}(\gamma + it)|^2 dt. \tag{1.22}$$

If $f \in L_2(\mathbb{R}_+; x^{2\gamma-1}dx)$, $g \in L_2(\mathbb{R}_+; x^{1-2\gamma}dx)$, then

$$\int_0^\infty f(x)g(x)dx = \frac{1}{2\pi} \int_{\gamma-i\infty}^{\gamma+i\infty} f^*(s)g^*(1-s)ds. \tag{1.23}$$

In particular, we have the following reciprocal Mellin transforms [2, 5, 7]

$$z^{\nu/2} K_{\nu/2}(2z) = \frac{1}{8\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \Gamma(w)\Gamma\left(\frac{w+\nu}{2}\right) z^{-w} dw, \ \gamma > \max(0, -\text{Re } \nu), \tag{1.24}$$

$$\int_0^\infty K_{\nu/2}(2t) t^{w+\nu/2-1} dt = \frac{1}{4}\Gamma(w)\Gamma\left(\frac{w+\nu}{2}\right), \ \text{Re } w > 0. \tag{1.25}$$

Further, by relation (2.16.33.10) in [5] we get the Mellin transform (1.10) of the kernel (1.1)

$$\begin{aligned}&\int_0^\infty x^{s-1} K_{i\tau}\left(\sqrt{x^2+y^2} - y\right) K_{i\tau}\left(\sqrt{x^2+y^2} + y\right) dx = \\ &= \frac{\sqrt{\pi}}{2} y^{s/2} K_{s/2}(2y) \frac{\Gamma\left(\frac{s}{2} + i\tau\right)\Gamma\left(\frac{s}{2} - i\tau\right)}{\Gamma((1+s)/2)},\end{aligned} \tag{1.26}$$

which is true for all $y, \tau > 0$ and $\gamma = \text{Re}s > 0$. Hence multiplying both sides of (1.26) by $y^{\omega-1}$, $\text{Re}\omega > 0$, we integrate with respect to $y > 0$. Then by using (1.25) we arrive at the value of the double Mellin transform [1, 4] for the kernel (1.1) as

$$\begin{aligned}&\int_0^\infty\int_0^\infty x^{s-1} y^{\omega-1} K_{i\tau}\left(\sqrt{x^2+y^2} - y\right) K_{i\tau}\left(\sqrt{x^2+y^2} + y\right) dxdy = \\ &= \frac{\sqrt{\pi}}{8}\Gamma(\omega)\Gamma\left(\frac{s+\omega}{2}\right) \frac{\Gamma\left(\frac{s}{2} + i\tau\right)\Gamma\left(\frac{s}{2} - i\tau\right)}{\Gamma((1+s)/2)}.\end{aligned} \tag{1.27}$$

So according to [1,4] the double Mellin transform

$$f^*(s,w) = \int_0^\infty \int_0^\infty f(x,y)\ x^{s-1} y^{w-1} dxdy,\ (s,w) \in \mathbb{C} \times \mathbb{C} \tag{1.28}$$

is well defined for any $f(x,y) \in L_1\left(\mathbb{R}_+ \times \mathbb{R}_+;\ x^{\gamma_1 - 1} y^{\gamma_2 - 1} dxdy\right),\ \gamma_1 = \text{Re}s, \gamma_2 = \text{Re}w$. When $f^*(s,w) \in L_1((\gamma_1 - i\infty, \gamma_1 + i\infty) \times (\gamma_2 - i\infty, \gamma_2 + i\infty))$, i.e.

$$\int_{\gamma_1 - i\infty}^{\gamma_1 + i\infty} \int_{\gamma_2 - i\infty}^{\gamma_2 + i\infty} |f^*(s,w)\ dsdw| < \infty,$$

then the inversion formula

$$f(x,y) = \frac{1}{(2\pi i)^2} \int_{\gamma_1 - i\infty}^{\gamma_1 + i\infty} \int_{\gamma_2 - i\infty}^{\gamma_2 + i\infty} f^*(s,w)\ x^{-s} y^{-w}\ dsdw \tag{1.29}$$

is true for all $(x,y) \in \mathbb{R}_+ \times \mathbb{R}_+$. Analogously to the one-dimensional case, the double Mellin transform (1.28) with the convergence in the mean square sense

$$f^* : L_2\left(\mathbb{R}_+ \times \mathbb{R}_+;\ x^{2\gamma_1 - 1} y^{2\gamma_2 - 1} dxdy\right) \leftrightarrow L_2((\gamma_1 - i\infty, \gamma_1 + i\infty) \times (\gamma_2 - i\infty,\ \gamma_2 + i\infty))$$

is an isometric isomorphism between these spaces (see [1]) and

$$\int_0^\infty \int_0^\infty |f(x,y)|^2 x^{2\gamma_1 - 1} y^{2\gamma_2 - 1} dxdy = \frac{1}{(2\pi)^2} \int_{-\infty}^\infty \int_{-\infty}^\infty |f^*(\gamma_1 + iu, \gamma_2 + iv)|^2 dudv. \tag{1.30}$$

More generally, for

$$f \in L_2\left(\mathbb{R}_+ \times \mathbb{R}_+; x^{2\gamma_1 - 1} y^{2\gamma_2 - 1} dxdy\right), \quad g \in L_2\left(\mathbb{R}_+ \times \mathbb{R}_+; x^{1-2\gamma_1} y^{1-2\gamma_2} dxdy\right)$$

it has

$$\int_0^\infty \int_0^\infty f(x,y) g(x,y) dxdy = \frac{1}{(2\pi i)^2} \int_{\gamma_1 - i\infty}^{\gamma_1 + i\infty} \int_{\gamma_2 - i\infty}^{\gamma_2 + i\infty} f^*(s,w) g^*(1-s, 1-w) dsdw. \tag{1.31}$$

2. AN INVERSION THEOREM

We define functions $G(x,y)$ in (1.1) belonging to a class of double Mellin integrals

$$G(x,y) = \frac{1}{(2\pi i)^2} \int_{\gamma - i\infty}^{\gamma + i\infty} \int_{\frac{1}{2} - i\infty}^{\frac{1}{2} + i\infty} \frac{\Gamma(\omega)\, \Gamma((s+\omega)/2)}{\Gamma((1+s)/2)} \varphi(s)\ x^{-s} y^{-w} dsdw, \tag{2.1}$$

where $x > 0$, $y > 0$, $0 < \gamma < 1$, $\varphi(s)$, $s = \gamma + it$ is analytic in the strip $|\text{Re}s| < 1$. Moreover, it belongs to the Hardy weighted space $H_1^{(-1,1)}(\mathbb{R}; (|t|+1)^{3/2}dt)$ satisfying the condition

$$\sup_{|\text{Re}s|<1} \int_{-\infty}^{\infty} |\varphi(\text{Re}s + it)| (|t|+1)^{3/2} dt < \infty. \tag{2.2}$$

First we observe that the integral (2.1) converges absolutely and uniformly for all $x \geq x_0$, $y \geq y_0$. In fact, with condition (2.2) and elementary inequality for Euler's beta-function $|B(a,b)| \leq B(\text{Re } a, \text{Re } b)$ [2, vol. I] we find from (2.1)

$$|G(x,y)| \leq \frac{1}{(2\pi)^2} \int_{\gamma-i\infty}^{\gamma+i\infty} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \left| \frac{\Gamma(\omega)\Gamma((s+\omega)/2)}{\Gamma((1+s)/2)} \varphi(s)\ x^{-s} y^{-w} ds dw \right| \leq$$

$$\leq \frac{x_0^{-\gamma} y_0^{-1/2}}{(2\pi)^2} \int_{\gamma-i\infty}^{\gamma+i\infty} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} |\Gamma(\omega)| \left| \frac{\Gamma((s+\omega)/2)}{\Gamma((1+s)/2)} \varphi(s) \right| |ds dw| =$$

$$= \frac{x_0^{-\gamma} y_0^{-1/2}}{(2\pi)^2} \int_{\gamma-i\infty}^{\gamma+i\infty} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \left| \frac{\Gamma(\omega)}{\Gamma((1-w)/2)} \right| |B((s+\omega)/2, (1-w)/2)| |\varphi(s)| |ds dw| <$$

$$< \frac{x_0^{-\gamma} y_0^{-1/2}}{(2\pi)^2} B((2\gamma+1)/4, 1/4) \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \left| \frac{\Gamma(\omega)}{\Gamma((1-w)/2)} dw \right| \times$$

$$\times \int_{\gamma-i\infty}^{\gamma+i\infty} |\varphi(s)| (|s|+1)^{3/2}\ |ds| < \infty.$$

Hence appealing to (1.24) we calculate the integral with respect to w in (2.1) and write it as follows

$$G(x,y) = \frac{2}{\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{y^{s/2} K_{s/2}(2y)}{\Gamma((1+s)/2)} \varphi(s)\ x^{-s} ds. \tag{2.3}$$

This integral is also absolutely convergent for each $x > 0$, $y > 0$. Indeed, with condition (2.2), asymptotic behavior (1.13) and asymptotic Stirling's formula for gamma-functions [2, vol. I] we obtain

$$\int_{\gamma-i\infty}^{\gamma+i\infty} \left| \frac{y^{s/2} K_{s/2}(2y)}{\Gamma((1+s)/2)} \varphi(s)\ x^{-s} ds \right| < C x^{-\gamma} y^{\gamma/2} \int_{\gamma-i\infty}^{\gamma+i\infty} |\varphi(s)| (|s|+1)^{-\gamma-1/2} |ds| <$$

$$< C x^{-\gamma} y^{\gamma/2} \int_{\gamma-i\infty}^{\gamma+i\infty} |\varphi(s)| (|s|+1)^{3/2} |ds| < \infty,$$

where $C > 0$ is an absolute constant. However, since $|\varphi\,(\mathrm{Re}s\ + it)\,|(|t| + 1)^{3/2}$ is bounded (see (2.2)), we return to (2.1) and easily verify, that its integrand belongs to $L_2((-\gamma - i\infty, -\gamma + i\infty) \times (1/2 - i\infty,\ 1/2 + i\infty))$, $|\gamma| < 1$. This means via (1.30), that $G(x, y) \in L_2\left(\mathbb{R}_+ \times \mathbb{R}_+;\ x^{-2\gamma-1}dxdy\right)$. Taking into account that the right-hand side of equality (1.27) belongs to $L_2((\gamma - i\infty, \gamma + i\infty) \times (1/2 - i\infty,\ 1/2 + i\infty))$ we apply (1.31) to write the double transformation (1.1) in the form

$$F(\tau) = \frac{1}{(2\pi i)^2}\frac{\sqrt{\pi}}{8}\int\limits_{\gamma-i\infty}^{\gamma+i\infty}\int\limits_{1/2-i\infty}^{1/2+i\infty}\Gamma\left(\frac{s}{2} + i\tau\right)\Gamma\left(\frac{s}{2} - i\tau\right)\frac{\Gamma\left(\omega\right)\Gamma\left((s+\omega)/2\right)}{\Gamma((1+s)/2)}\times$$
$$\times\frac{\Gamma\left(1-\omega\right)\Gamma\left((1-\omega-s)/2\right)}{\Gamma((1-s)/2)}\varphi(-s)dsdw. \tag{2.4}$$

Meanwhile, the inner integral with respect to w in (2.4) equals (see (1.23), (1.24), relation (2.16.33.2) in [5])

$$\frac{1}{2\pi i}\int\limits_{1/2-i\infty}^{1/2+i\infty}\Gamma\left(\omega\right)\Gamma\left((s+\omega)/2\right)\Gamma\left(1-\omega\right)\Gamma\left((1-\omega-s)/2\right)dw =$$

$$= 16\int\limits_0^\infty K^2_{s/2}(2v)dv = 2\pi\ \Gamma\left(\frac{1+s}{2}\right)\Gamma\left(\frac{1-s}{2}\right).$$

Therefore, similarly to (2.3), after calculation with respect to w we write (2.4) in the form

$$F(\tau) = \frac{\sqrt{\pi}}{8i}\int\limits_{\gamma-i\infty}^{\gamma+i\infty}\Gamma\left(\frac{s}{2} + i\tau\right)\Gamma\left(\frac{s}{2} - i\tau\right)\varphi(-s)ds. \tag{2.5}$$

In order to continue our consideration we will first investigate the following index integral (see (1.4))

$$I(\alpha, s) = \int\limits_0^\infty \tau\sinh\alpha\tau\cosh\pi\tau K_{i\tau}\left(\sqrt{x^2+y^2} - y\right)K_{i\tau}\left(\sqrt{x^2+y^2} + y\right)\times$$
$$\times\Gamma\left(\frac{s}{2} + i\tau\right)\Gamma\left(\frac{s}{2} - i\tau\right)d\tau, \quad 0 \le \alpha \le \pi,\ 0 < \mathrm{Re}s < 1 \tag{2.6}$$

for any fixed $(x, y) \in \mathbb{R}_+ \times \mathbb{R}_+$. Taking into account (1.18) and Abel's test we observe that (2.6) converges uniformly with respect to $\alpha \in [0, \pi]$. When $0 \le \alpha < \pi$ we employ representation (1.19), asymptotic behavior by index (1.13) of the Macdonald function and Fubini's theorem to obtain

$$I(\alpha, s) = \frac{\sqrt{\pi}2^{s/2}}{\Gamma((1-s)/2)}\int\limits_0^\infty\int\limits_0^\infty \tau\sinh\alpha\tau\ t^{s/2-1}e^t K_{i\tau}(t)\times$$
$$\times K_{i\tau}\left(\sqrt{x^2+y^2} - y\right)K_{i\tau}\left(\sqrt{x^2+y^2} + y\right)d\tau\ dt. \tag{2.7}$$

Hence (1.14) and Fubini's theorem yield

$$I(\alpha, s) = \frac{\sqrt{\pi}\,\Gamma(s/2)}{\Gamma((1-s)/2)} \int_0^\infty \int_0^\infty \frac{\tau \sinh \alpha\tau \cos u\tau}{\sinh^s(u/2)} \times \\ \times K_{i\tau}\left(\sqrt{x^2+y^2} - y\right) K_{i\tau}\left(\sqrt{x^2+y^2} + y\right) d\tau\, du. \tag{2.8}$$

Denoting by $a = \sqrt{x^2+y^2} - y$, $b = \sqrt{x^2+y^2} + y$, which are fixed numbers and integrating by parts in the inner integral with respect to u, we rewrite (2.8) in the form

$$I(\alpha, s) = \frac{\sqrt{\pi}\,\Gamma(1+s/2)}{\Gamma((1-s)/2)} \int_0^\infty \sinh \alpha\tau K_{i\tau}(a) K_{i\tau}(b) h(\tau, s) d\tau, \tag{2.9}$$

where

$$h(\tau, s) = \int_0^\infty \frac{\sin u\tau \cosh(u/2)}{\sinh^{s+1}(u/2)} du, \quad 0 < \gamma = \mathrm{Re} s < 1. \tag{2.10}$$

Moreover, the integral (2.10) converges absolutely and uniformly with respect to s in the strip $\{s = \gamma + it;\ 0 \le \gamma < 1,\ t \in \mathbb{R}\}$ and with respect to τ on any compact in $\mathbb{R}_+$. Furthermore, we have the estimate

$$|h(\tau, s)| = \tau^\gamma \left| \int_0^\infty \left(\frac{u/\tau}{\sinh(u/(2\tau))} \right)^{s+1} \cosh(u/(2\tau)) \frac{\sin u}{u^{s+1}} du \right| \le \\ \le C_\gamma \tau^\gamma \int_0^\infty \frac{|\sin u|}{u^{\gamma+1}} du = O(\tau^\gamma),\ \tau \to \infty,\ 0 < \gamma < 1, \tag{2.11}$$

where $C_\gamma > 0$ is a constant, which is not depending on τ, since the function

$$\xi_s(v) = \begin{cases} \cosh(v/2) \left(\frac{v}{\sinh(v/2)} \right)^{s+1}, & \text{if } v \ne 0, \\ 2^{s+1}, & \text{if } v = 0, \end{cases}$$

is bounded for all $v \ge 0$, $0 < \mathrm{Re} s < 1$. Further, expanding $\xi_s(v)$ in a Taylor series near zero we find $\xi(v) = 2^{s+1} + O(v^2)$, $0 \le v \le 1$. Therefore, (2.10) becomes

$$h(\tau, s) = \tau^\gamma \int_0^1 \xi_\gamma \left(\frac{u}{\tau} \right) \frac{\sin u}{u^{s+1}} \left(\frac{u/\tau}{\sinh(u/(2\tau))} \right)^{it+1} du + \\ + \int_1^\infty \frac{\sin u\tau \cosh(u/2)}{\sinh^{s+1}(u/2)} du = O(\tau^\gamma) + O\left(\tau^{\gamma-2}\right) + O(1), \quad \tau \to +\infty \tag{2.12}$$

uniformly by s in the strip $\{s = \gamma + it;\ 0 < \gamma < 1,\ t \in \mathbb{R}\}$. Hence choosing a sufficiently big and fixed number $A > 0$ we split the integral (2.9) into $\int_0^A,\ \int_A^\infty$. Denoting by

$$I_1(\alpha, s) = \frac{\sqrt{\pi}\ \Gamma(1 + s/2)}{\Gamma((1 - s)/2)} \int_0^A \sinh \alpha\tau K_{i\tau}(a) K_{i\tau}(b) h(\tau, s) d\tau$$

we observe, appealing again to the Stirling asymptotic formula for gamma-functions, that the following estimate is true

$$|I_1(\alpha, s)| < \sqrt{\pi} \left| \frac{\Gamma(1 + s/2)}{\Gamma((1 - s)/2)} \right| \int_0^A \tau \sinh \alpha\tau |K_{i\tau}(a) K_{i\tau}(b)| d\tau \times$$

$$\times \int_0^\infty \frac{u \cosh(u/2)}{\sinh^{\gamma+1}(u/2)} du < C_{a,b,\gamma} |s|^{\gamma+1/2} = O\left(|s|^{\gamma+1/2}\right), \quad |s| \to \infty,$$

where $C_{a,b,\gamma} > 0$ is a constant, which does not depend on $\alpha \in [0, \pi]$. Meanwhile, the integral

$$I_2(\alpha, s) = \frac{\sqrt{\pi}\ \Gamma(1 + s/2)}{\Gamma((1 - s)/2)} \int_A^\infty \sinh \alpha\tau K_{i\tau}(a) K_{i\tau}(b) h(\tau, s) d\tau$$

can be treated by (2.12) and the asymptotic formula (1.13). So, we have

$$I_2(\alpha, s) = \frac{\pi\sqrt{2}\ \Gamma(1 + s/2)}{\Gamma((1 - s)/2)} \int_A^\infty e^{-\pi\tau} \frac{\sinh \alpha\tau}{\tau} h(\tau, s) \sin\left(\tau\left(\log \frac{2\tau}{a} - 1\right) + \frac{\pi}{4} + \frac{a^2}{4\tau}\right) \times$$

$$\times \sin\left(\tau\left(\log \frac{2\tau}{b} - 1\right) + \frac{\pi}{4} + \frac{b^2}{4\tau}\right)\left(1 + O\left(\frac{1}{\tau}\right)\right)^2 d\tau =$$

$$= \frac{\pi\ \Gamma(1 + s/2)}{\sqrt{2}\Gamma((1 - s)/2)} \int_A^\infty e^{-\pi\tau} \sinh \alpha\tau \frac{h(\tau, s)}{\tau} \times$$

$$\times \cos\left(\tau \log \frac{b}{a} + \frac{(a^2 - b^2)}{4\tau}\right)\left(1 + O\left(\frac{1}{\tau}\right)\right)^2 d\tau +$$

$$+ \frac{\pi\ \Gamma(1 + s/2)}{\sqrt{2}\Gamma((1 - s)/2)} \int_A^\infty e^{-\pi\tau} \sinh \alpha\tau \frac{h(\tau, s)}{\tau} \times$$

$$\times \sin\left(2\tau\left(\log \frac{2\tau}{\sqrt{ab}} - 1\right) + \frac{(a^2 + b^2)}{4\tau}\right)\left(1 + O\left(\frac{1}{\tau}\right)\right)^2 d\tau =$$

$$= J_1(\alpha, s) + J_2(\alpha, s).$$

Hence by using the second mean value theorem, the estimate (2.12), the Schwarz inequality, the Parseval equality for the sine Fourier transform and the Dirichlet convergence test for integrals we derive the uniform estimate by $\alpha \in [0, \pi]$ and $s = \gamma + it,\ 0 < \gamma < 1,\ t \in \mathbb{R}$, namely

$$|J_1(\alpha, s)| < e^{(\alpha-\pi)A} \left| \frac{\Gamma(1+s/2)}{\Gamma((1-s)/2)} \right| \left[O\left(\int_A^{A_1} (\tau^{\gamma-1} + \tau^{\gamma-3}) \cos\left(\tau \log \frac{b}{a} + \frac{(a^2-b^2)}{4\tau} \right) d\tau \right) + \right.$$

$$\left. + \text{ const.} \int_A^{A_1} \frac{d\tau}{\tau} \left| \int_1^\infty \frac{\sin u\tau \cosh(u/2)}{\sinh^{s+1}(u/2)} du \right| \right] <$$

$$< \left| \frac{\Gamma(1+s/2)}{\Gamma((1-s)/2)} \right| \left[O(1) + \text{const.} \left(\int_1^\infty \frac{\cosh^2(u/2)}{\sinh^{2(\gamma+1)}(u/2)} du \right)^{1/2} \right] =$$

$$= O\left(|s|^{\gamma+1/2} \right).$$

We similarly treat $J_2(\alpha, s)$ to establish the uniform relation $J_2(\alpha, s) = O\left(|s|^{\gamma+1/2}\right)$ by $[0, \pi]$. Thus returning to (2.6), (2.8), (2.9) and taking into account the above discussions we prove that $I(\alpha, s) = O\left(|s|^{\gamma+1/2}\right)$ uniformly by $\alpha \in [0, \pi]$ and $s = \gamma + it$, $0 < \gamma < 1$, $t \in \mathbb{R}$.

Multiplying both sides of (2.5) by

$$\frac{2^5}{\pi^4} \tau \sinh \alpha\tau \cosh \pi\tau K_{i\tau}\left(\sqrt{x^2+y^2} - y \right) K_{i\tau}\left(\sqrt{x^2+y^2} + y \right)$$

and integrating through with respect to τ over $\mathbb{R}_+$ we change the order of integration by Fubini's theorem for each $\alpha \in [0, \pi)$. Then we pass to the limit $\alpha \to \pi-$ due to the obtained estimates, condition (2.2) and the Lebesgue dominated convergence theorem. Hence the uniform convergence of the integral (2.6) and relation (1.18) lead us to the formula

$$\frac{2^5}{\pi^4} \lim_{\alpha \to \pi-} \int_0^\infty \tau \sinh \alpha\tau \cosh \pi\tau K_{i\tau}\left(\sqrt{x^2+y^2} - y \right) K_{i\tau}\left(\sqrt{x^2+y^2} + y \right) F(\tau) d\tau =$$
$$= \frac{2}{\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{y^{-s/2} K_{s/2}(2y)}{\Gamma((1-s)/2)} \varphi(-s) x^s ds. \tag{2.13}$$

Our goal is to prove that the right-hand side of (2.13) is equal to $G(x, y),\ x, y > 0$ and we will arrive at the inversion formula (1.4). In fact, by virtue of (2.3), the analyticity of the integrand in the strip $|\text{Re}s| < 1$ and Cauchy's theorem we have

$$\frac{2}{\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{y^{-s/2}K_{s/2}(2y)}{\Gamma((1-s)/2)}\varphi(-s)x^{s}ds = \frac{2}{\pi i}\int_{-\gamma-i\infty}^{-\gamma+i\infty}\frac{y^{s/2}K_{s/2}(2y)}{\Gamma((1+s)/2)}\varphi(s)x^{-s}ds =$$

$$= \frac{2}{\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{y^{s/2}K_{s/2}(2y)}{\Gamma((1+s)/2)}\varphi(s)x^{-s}ds = G(x,y)$$

since for all $x, y > 0$

$$\lim_{B\to\infty}\int_{\pm\gamma\pm iB}^{\mp\gamma\pm iB}\frac{y^{s/2}K_{s/2}(2y)}{\Gamma((1+s)/2)}\varphi(s)x^{-s}ds = 0$$

via condition (2.2). Thus we summarize our results by the following

Theorem 2.1. *Under condition* (2.2) *the* (1.1) *is well-defined in the class of double Mellin integrals* (2.1) *and the inversion formula* (1.4) *holds for all* $x, y > 0$, *where the convergence in* α *is pointwise.*

Corollary 2.2. *Under the conditions of Theorem 2.1 the transformation* (1.1) *is a bounded continuous function on* $[0,\infty)$ *and behaves as* $O\left(\tau^{\gamma-1}e^{-\pi\tau}\right)$, $\tau \to +\infty$, $0 < \gamma < 1$.

Proof. In fact (1.1) can be written as an absolutely and uniformly convergent integral

$$F(\tau) = \int_0^\infty\int_0^\infty K_{i\tau}\left(\sqrt{x^2+y^2}-y\right)K_{i\tau}\left(\sqrt{x^2+y^2}+y\right)G(x,y)\frac{dxdy}{x}$$

since the Macdonald function satisfies the inequality $|K_z(x)| \le K_{\text{Re}z}(x)$, $x > 0$ and therefore for all $\tau \ge 0$

$$|F(\tau)| \le \int_0^\infty\int_0^\infty K_0\left(\sqrt{x^2+y^2}-y\right)K_0\left(\sqrt{x^2+y^2}+y\right)|G(x,y)|\frac{dxdy}{x} < \infty.$$

Hence returning to the representation (2.5) and employing (1.14), (1.19), (2.10) we derive

$$\tau^{1-\gamma}\cosh(\pi\tau)F(\tau) = \frac{\tau^{1-\gamma}\pi}{8i}\int_{\gamma-i\infty}^{\gamma+i\infty}\int_0^\infty t^{s/2-1}e^{-t}K_{i\tau}(t)\frac{\varphi(-s)2^{s/2}}{\Gamma((1-s)/2)}dtds =$$

$$= \frac{\tau^{-\gamma}\pi}{8i}\int_{\gamma-i\infty}^{\gamma+i\infty}\varphi(-s)\frac{\Gamma(1+s/2)}{\Gamma((1-s)/2)}h(\tau,s)ds.$$

Consequently (see (2.2), (2.11)),

$$\tau^{1-\gamma}\cosh(\pi\tau)|F(\tau)| \le \frac{\tau^{-\gamma}\pi}{8}\int_{\gamma-i\infty}^{\gamma+i\infty}\left|\varphi(-s)\frac{\Gamma(1+s/2)}{\Gamma((1-s)/2)}h(\tau,s)ds\right| <$$

$$< C_\gamma \sup_{|\mathrm{Re}s|<1}\int_{-\infty}^{\infty}|\varphi\left(\mathrm{Re}s+it\right)|(|t|+1)^{3/2}dt = O(1)$$

and we prove Corollary 2.2. □

3. INTEGRAL EQUATIONS OF THE LEBEDEV TYPE

In this section we will apply the inversion theorem for the transformation (1.1) to find particular solutions of integral equation (1.5) and its particular cases in the class of double integrals (2.1). Precisely, we consider equation (1.5), where the kernel $S(\tau,t)$ is represented by the Kontorovich-Lebedev integral (1.7)

$$S(\tau,t) = \int_0^\infty K_{i\tau}(u)\psi(u,t)du, \tag{3.1}$$

where $\psi(u,t)\neq 0$, $(u,t)\in\mathbb{R}_+\times\mathbb{R}_+$ is a continuous function. Substituting (3.1) into (1.5) and making change of variables $u=\sqrt{x^2+y^2}-y$, $t=\sqrt{x^2+y^2}+y$ we write this equation in terms of the double integral (1.1)

$$\int_0^\infty\int_0^\infty K_{i\tau}\left(\sqrt{x^2+y^2}-y\right)K_{i\tau}\left(\sqrt{x^2+y^2}+y\right)\times$$

$$\times\psi\left(\sqrt{x^2+y^2}-y,\ \sqrt{x^2+y^2}+y\right)f\left(\sqrt{x^2+y^2}+y\right)\frac{2x}{\sqrt{x^2+y^2}}dxdy = F(\tau), \tag{3.2}$$

letting

$$G(x,y) = \psi\left(\sqrt{x^2+y^2}-y,\ \sqrt{x^2+y^2}+y\right)f\left(\sqrt{x^2+y^2}+y\right)\frac{2x^2}{\sqrt{x^2+y^2}}, \tag{3.3}$$

which is, in turn, being represented by (2.1), (2.3). So, writing $t=\sqrt{x^2+y^2}+y=r(1+\sin\lambda)$, $u=\sqrt{x^2+y^2}-y=r(1-\sin\lambda)$, $r>0$, $\lambda\in[0,\pi/2)$, and letting $\rho=\frac{1-\sin\lambda}{1+\sin\lambda}$, $z=r(1+\sin\lambda)$ we seek a desired solution of equation (1.5) in the form (cf. (2.3))

$$f(z) = \frac{(1+\rho)}{2z\rho\ \psi(\rho z,z)\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{K_{s/2}(z(1-\rho))}{\Gamma((1+s)/2)}\varphi(s)\ \left(\frac{2\rho z}{1-\rho}\right)^{-s/2}ds$$

under the conditions of Theorem 2.1. Then from (1.4) it has a family of solutions

$$f(t) = \frac{4(1+\rho)}{\pi^4\psi(\rho t,t)\rho}\int_0^\infty \tau \sinh 2\pi\tau K_{i\tau}(t)K_{i\tau}(\rho t)F(\tau)d\tau,\ t>0, \tag{3.4}$$

for any $\rho \in (0,1]$, where the convergence of the integral (3.4) is understood by (1.4).

As an example let $\psi(u,t) \equiv 1$ in (3.1). Then via relation (2.16.2.1) in [5] we get $S(\tau,t) = \frac{\pi}{2}[\cosh(\pi\tau/2)]^{-1}$. Hence the corresponding integral equation (1.5) takes the form of the modified Kontorovich-Lebedev integral equation (see (1.7))

$$\frac{\pi}{2\cosh(\pi\tau/2)}\int_0^\infty K_{i\tau}(t)f(t)dt = F(\tau),$$

which has a solution

$$f(t) = \frac{4(1+\rho)}{\pi^4\rho}\int_0^\infty \tau \sinh 2\pi\tau K_{i\tau}(t)K_{i\tau}(\rho t)F(\tau)d\tau,\ t>0, \tag{3.5}$$

for any $\rho \in (0,1]$ in the class (2.1), where

$$G(x,y) = f\left(\sqrt{x^2+y^2}+y\right)\frac{2x^2}{\sqrt{x^2+y^2}}.$$

Combining with the direct Mellin transform (1.28) we derive the equality

$$2\int_0^\infty\int_0^\infty \frac{f\left(\sqrt{x^2+y^2}+y\right)}{\sqrt{x^2+y^2}}\, x^{s+1}y^{w-1}dxdy = \frac{\Gamma(\omega)\,\Gamma((s+\omega)/2)}{\Gamma((1+s)/2)}\varphi(s). \tag{3.6}$$

Hence (see (1.20)) with the change of variables the left-hand side of (3.6) is equivalent to relations

$$2\int_0^\infty\int_0^\infty \frac{f\left(\sqrt{x^2+y^2}+y\right)}{\sqrt{x^2+y^2}} x^{s+1}y^{w-1}dxdy = 2f^*(s+w+1)\int_0^{\pi/2}\frac{\cos^{s+1}\lambda\sin^{w-1}\lambda}{(1+\sin\lambda)^{s+w+1}}d\lambda =$$

$$= 2^{1-w}f^*(s+w+1)\frac{\Gamma(\omega)\,\Gamma\left(\frac{s}{2}+1\right)}{\Gamma(w+\frac{s}{2}+1)}.$$

Therefore from (3.6) and the duplication formula for gamma-functions [2, vol. I] we obtain that the Mellin transform of f satisfies the following condition

$$f^*(s+w+1) = \frac{2^{w+s-1}}{\sqrt{\pi}}\Gamma\left(w+\frac{s}{2}+1\right)\Gamma\left(\frac{s+\omega}{2}\right)\frac{\varphi(s)}{\Gamma(1+s)}. \tag{3.7}$$

Denoting by

$$g(z) = \frac{\sqrt{\pi} f^*(z+1)}{2^{z-1}\Gamma\left(\frac{z}{2}\right)},$$

we apply the reduction formula for gamma-functions [2, vol. I] to write finally from (3.7) a functional equation for $g(z)$

$$g(s+w) = \left(w + \frac{s}{2}\right) g(s+w-1).$$

The Lebedev equation (1.8) can be treated employing relation (2.16.52.10) in [5] and an inversion formula of the Kontorovich-Lebedev transform [6, 7]. As a result we get the following equation for the modified Bessel function

$$\frac{\pi}{\cosh \pi\tau} K_{i\tau}(t) = \sqrt{t} \int_0^\infty \frac{K_{i\tau}(u)}{\sqrt{u}} \frac{e^{-u-t}}{u+t} du,$$

which transforms the left-hand side of (1.8) after the change of variables (see above) into the double integral

$$\int_0^\infty \int_0^\infty K_{i\tau}\left(\sqrt{x^2+y^2} - y\right) K_{i\tau}\left(\sqrt{x^2+y^2} + y\right) \frac{e^{-2\sqrt{x^2+y^2}}}{x^2+y^2} \times$$

$$\times \sqrt{\frac{\sqrt{x^2+y^2}+y}{\sqrt{x^2+y^2}-y}}\, f\left(\sqrt{x^2+y^2}+y\right) x \, dxdy = F(\tau).$$

Hence it has a solution

$$f(t) = \frac{4(1+\rho)^2 t}{\pi^4 \sqrt{\rho}} e^{t(1+\rho)} \int_0^\infty \tau \sinh 2\pi\tau K_{i\tau}(t) K_{i\tau}(\rho t) F(\tau) d\tau, \ t > 0,$$

for any $\rho \in (0, 1]$ in the class (2.1), where

$$G(x, y) = \frac{e^{-2\sqrt{x^2+y^2}}}{x^2+y^2} \sqrt{\frac{\sqrt{x^2+y^2}+y}{\sqrt{x^2+y^2}-y}}\, f\left(\sqrt{x^2+y^2}+y\right) x^2.$$

Furthermore, making substitutions in polar coordinates, condition (3.6) for this case takes the form

$$\int_0^\infty \int_0^{\pi/2} e^{-2r(1+\sin\lambda)^{-1}} \frac{\cos^{s+1}\lambda \sin^{w-1}\lambda}{\sqrt{1-\sin\lambda}(1+\sin\lambda)^{s+w-1/2}} f(r)\, r^{s+w-1} dr d\lambda =$$

$$= \frac{\Gamma(\omega)\Gamma((s+\omega)/2)}{\Gamma((1+s)/2)} \varphi(s). \tag{3.8}$$

Integration by λ in (3.8) leads to the equality (cf. (3.7))

$$\int_0^\infty e^{-r}\Phi_1\left(1+\frac{s}{2},1,1+\frac{s}{2}+w,-1,r\right) f(r)\, r^{s+w-1}dr =$$

$$= \frac{2^{s+w-1}}{\sqrt{\pi}} \frac{\Gamma\left(1+\omega+\frac{s}{2}\right)\Gamma\left((s+\omega)/2\right)}{\Gamma(1+s)}\varphi(s),$$

where $\Phi_1(a,b,c,x,y)$ is a hypergeometric function of two variables from the Horn list [2, vol. I].

Finally we consider equation (1.6). We treat it with the use of the representation (1.15). It takes the form (1.5), where $\psi(u,t)=\frac{1}{2u}e^{-u-\frac{t^2}{2u}}$ for the kernel (3.1) $S(\tau,t)=K^2_{i\tau}(t)$. Thus from (3.4) we find a family of its solutions

$$f(t) = 8(1+\rho)e^{t\left(\rho+\frac{1}{2\rho}\right)}\frac{t}{\pi^4}\int_0^\infty \tau\sinh 2\pi\tau K_{i\tau}(t)K_{i\tau}(\rho t)F(\tau)d\tau,\ \rho\in(0,1],$$

and the corresponding equality (3.6) will be in the form

$$\int_0^\infty\int_0^1 e^{-r\left(x+\frac{1}{2x}\right)}f(r)(r(1-x))^{\omega-1}(r\sqrt{x})^s\frac{dxdr}{x} = \frac{2^\omega\Gamma(\omega)\Gamma((s+\omega)/2)}{\Gamma((1+s)/2)}\varphi(s).$$

Acknowledgements

The present investigation was supported, in part, by the "Centro de Matemática" of the University of Porto.

REFERENCES

[1] Yu.A. Brychkov, H-J.Glaeske, A.P.Prudnikov, V.K. Tuan, *Multidimensional Integral Transformations*, Gordon and Breach, Philadelphia, Paris, Tokyo, Melbourne, 1992.

[2] A. Erdélyi, W. Magnus, F. Oberhettinger, F.G. Tricomi, *Higher Transcendental Functions*, vols. I, II, McGraw-Hill, New York, London and Toronto, 1953.

[3] N.N. Lebedev, *On the representation of an arbitrary function by the integrals involving square of the Macdonald functions with the imaginary index*, Sib. Mat. Journ. **3** (1962), 213–222.

[4] N.T. Hai, S.B. Yakubovich, *The Double Mellin-Barnes Type Integrals and Their Applications to Convolution Theory*, World Scientific, Singapore, 1992.

[5] A.P. Prudnikov, Yu.A. Brychkov, O.I. Marichev, *Integrals and Series: Special Functions*, Gordon and Breach, New York, 1986.

[6] I.N. Sneddon, *The Use of Integral Transforms*, McGray Hill, New York, 1972.

[7] S.B. Yakubovich, *Index Transforms*, World Scientific, Singapore, 1996.

[8] S.B. Yakubovich, *On a new index transformation related to the product of Macdonald functions*, Radovi Mat. **13** (2004), 63–85.

[9] S.B. Yakubovich, *The Plancherel and Hausdorff-Young type theorems for an index transformation*, Journ. for Anal. and Appl. **25** (2006), 193–204.

[10] S.B. Yakubovich, *The Kontorovich-Lebedev transformation on Sobolev type spaces*, Sarajevo Journ. of Math. **1** (2005) 14, 211–234.

Semyon B. Yakubovich
syakubov@fc.up.pt

University of Porto
Department of Pure Mathematics, Faculty of Science
Campo Alegre st., 687
4169-007 Porto, Portugal

Received: June 8, 2009.
Accepted: June 13, 2009.