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# EXTENSIONS OF SOLUTIONS OF A FUNCTIONAL EQUATION IN TWO VARIABLES

**Abstract.** An extension theorem for the functional equation of several variables

$$f(M(x,y)) = N(f(x), f(y)),$$

where the given functions $M$ and $N$ are left-side autodistributive, is presented.

**Keywords:** functional equation, autodistributivity, strict mean, extension theorem.

**Mathematics Subject Classification:** Primary 39B22.

## 1. AN EXTENSION THEOREM

Recently Zs. Páles [2] considered the extension problem for the functional equation of the form

$$f(x) = M(f(m_1(x,y)), \dots, f(m_k(x,y)))$$

where $M$ is a $k$-variable bisymmetric operation and $m_1, \dots, m_k$ some binary commuting operations.

In this note we deal with the extension problem for the functional equation

$$f(M(x,y)) = N(f(x), f(y))$$

where the given functions $M$ and $N$ are left-side autodistributive and $M$ is a strict mean. The functions satisfying this functional equation are called $(M,N)$-affine (cf. [1] where the case when both $M$ and $N$ are means is considered).

We prove the following

**Theorem 1.1.** *Let $I, J \subset \mathbb{R}$ be open intervals. Suppose that $M : I^2 \to I$, $N : J^2 \to J$ are continuous strictly increasing with respect to the first variable and such that*

$$M(M(x,y),z)) = M(M(x,z), M(y,z)), \quad x, y, z \in I, \tag{1.1}$$

$$N(N(x,y),z)) = N(N(x,z),N(y,z)), \quad x,y,z \in J. \tag{1.2}$$

*We also assume that $M$ is a strict mean, that is*

$$\min(x,y) < M(x,y) < \max(x,y), \quad x,y \in I,\ x \neq y.$$

*If $f : I_0 \to J$ satisfies the functional equation*

$$f\left(M(x,y)\right) = N(f(x),f(y)), \quad x,y \in I_0, \tag{1.3}$$

*for a nontrivial interval $I_0 \subset I$, then there exists a unique function $F : I \to J$ such that $F\left|_{I_0}\right. = f$ and*

$$F\left(M(x,y)\right) = N(F(x),F(y)), \quad x,y \in I.$$

*Proof.* Assume that $I_0 \subset I$ is a maximal subinterval of $I$ on which the function $f$ can be extended to satisfy equation (1.3). Suppose first that

$$b := \sup I_0 < \sup I.$$

Take an arbitrary $a \in I_0$, $a < b$. Then

$$f\left(M(x,y)\right) = N(f(x),f(y)), \quad x,y \in [a,b]. \tag{1.4}$$

Since $M$ is a continuous and strict mean, there is a $c \in I$, $c > b$ such that $M(a,c) < b$. Hence, as $M$ is strictly increasing with respect to the first variable,

$$M(x,c) < b, \quad x \in [a,c].$$

Setting $y := a$ in (1.4) we have

$$f(M(x,a)) = N(f(x),f(a)), \quad x \in [a,c].$$

Let $N_{f(a)}^{-1}$ denote the inverse function of $N(\cdot,f(a))$. Define $F_a : [a,c] \to J$ by

$$F_a(x) := N_{f(a)}^{-1}\left(f(M(x,a))\right), \quad x \in [a,c].$$

Note that by (1.4) the function $F_a$ is correctly defined,

$$F_a(x) = f(x), \quad x \in [a,b],$$

and

$$f(M(x,a)) = N(F_a(x),f(a)), \quad x \in [a,c]. \tag{1.5}$$

Now making use respectively of the definition of $F_a$, the property (1.1) of $M$, equation (1.4), equation (1.5), property (1.2) of $N$, for all $x,y \in [a,c]$ we have

$$\begin{aligned} F_a\left(M(x,y)\right) &:= N_{f(a)}^{-1}\left(f(M(M(x,y),a))\right) = N_{f(a)}^{-1}\left(f(M(M(x,a),M(y,a))\right) = \\ &= N_{f(a)}^{-1}\left(N\left(f(M(x,a)),f(M(y,a))\right)\right) = \\ &= N_{f(a)}^{-1}\left(N\left(N(F_a(x),f(a)),N(F_a(y),f(a))\right)\right) = \\ &= N_{f(a)}^{-1}(N(N(F_a(x),F_a(y)),f(a))) = N(F_a(x),F_a(y)), \end{aligned}$$

that is

$$F_a\left(M(x,y)\right) = N(F_a(x), F_a(y)), \quad x, y \in [a, c]. \tag{1.6}$$

In the case when $\inf I_0 \in I_0$ we can take $a = \inf I_0$. Putting $F := F_a$ we get

$$F\left(M(x,y)\right) = N(F(x), F(y)), \quad x, y \in [a, c].$$

Since $F\left|_{I_0}\right. = f$ and $I_0 \subsetneq [a, c]$, this contradicts the maximality of the interval $I_0$.

In the case when $\inf I_0 \notin I$, we take a decreasing sequence $(a_n : n \in \mathbb{N})$ such that $\inf I_0 = \lim_{n\to\infty} a_n$ and define $F : (\inf I_0, c] \to J$ by

$$F(x) := F_{a_n}(x), \quad x \in [a_n, c],\ n \in \mathbb{N}.$$

This definition is correct because, by (1.3),

$$m < n \Longrightarrow F_{a_m} = F_{a_n}\left|_{[a_m,c]}\right. .$$

In view of (1.6), we have

$$F\left(M(x,y)\right) = N(F(x), F(y)), \quad x, y \in (\inf I_0, c].$$

Since $F\left|_{I_0}\right. = f$ and $I_0 \subsetneq (\inf I_0, c]$, this contradicts the maximality of the interval $I_0$.

The obtained contradiction proves that $\sup I_0 = \sup I$. In a similar way we can show that $\inf I_0 = \inf I$. This completes the proof. □

## REFERENCES

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