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TOPOLOGICAL CLASSIFICATION OF CONFORMAL ACTIONS ON p-HYPERELLIPTIC AND (q,n)-GONAL RIEMANN SURFACES

Abstract. A compact Riemann surface X of genus $g > 1$ is said to be p-hyperelliptic if X admits a conformal involution ρ for which X/ρ has genus p. A conformal automorphism δ of prime order n such that X/δ has genus q is called a (q,n)-gonal automorphism. Here we study conformal actions on p-hyperelliptic Riemann surface with (q,n)-gonal automorphism.

Keywords: p-hyperelliptic Riemann surface, automorphism of a Riemann surface.

Mathematics Subject Classification: Primary: 30F20, 30F50; Secondary: 14H37, 20H30, 20H10.

1. INTRODUCTION

A compact Riemann surface X of genus $g \geq 2$ is said to be *p-hyperelliptic* if X admits a conformal involution ρ, called a p-hyperelliptic involution, such that X/ρ is an orbifold of genus p. This notion has been introduced by H. Farkas and I. Kra in [17] where they also proved that for $g > 4p+1$, p-hyperelliptic involution is unique and central in the group of all automorphisms of X. In [22] it has been proved that every two p-hyperelliptic involutions commute for $3p+2 \leq g \leq 4p+1$ and X admits at most two such involutions if $g > 3p+1$.

In the particular cases $p = 0, 1$, X are called *hyperelliptic* and *elliptic-hyperelliptic* Riemann surfaces respectively. Hyperelliptic Riemann surfaces and their automorphisms have received a good deal of attention in the literature. In [2] and [12] the authors determined the full groups of conformal automorphisms of such surfaces which made it possible to classify symmetry types of such actions in [5]. The p-hyperelliptic $(p \geq 1)$ surfaces at large have been studied in [7–11, 13–15] and [23], where the most attention has been paid to a study of groups of automorphisms of such surfaces and their symmetries.

We say that a finite group G acts on a topological surface X if there exists a monomorphism $\varepsilon : G \to \mathrm{Hom}^+(X)$, where $\mathrm{Hom}^+(X)$ is the group of orientation-preserving homeomorphisms of X. Two actions of finite groups G and G' on X are topological equivalent if the images of G and G' are conjugate in $\mathrm{Hom}^+(X)$. There are two reasons for the topological classification of finite actions rather than just the groups of homeomorphisms. First, the equivalence classes of group actions are in $1-1$ correspondence to conjugacy classes of finite subgroups of the mapping class group and so such a classification gives some information on the structure of this group. Second, the enumeration of finite group actions is a principal component of the analysis of singularities of the moduli space of conformal equivalence classes of Riemann surfaces of a given genus since such space is an orbit space of Teichmüller space by a natural action of the mapping class group, see [4].

The classification of conformal actions up to topological conjugacy is a classical problem, which has been considered for surfaces of genera $g = 2, 3$ in [3] and $g = 4$ in [1]. In the case p-hyperelliptic Riemann surfaces it has been studied in [24, 20] and [21] for $p = 0, 1$ and 2, respectively.

Here we study conformal actions on p-hyperelliptic Riemann surface X which admits a conformal automorphism δ of prime order $n > 2$ such that X/δ has genus q [18]. The automorphism δ is called the (q, n)-*gonal automorphism* and in the case $q = 0$, n-*gonality automorphism*. (q, n)-gonal Klein surfaces have been considered in [16].

2. PRELIMINARIES

We shall approach the problem using Riemann uniformization theorem by which each compact Riemann surface X of genus $g \geq 2$ can be represented as the orbit space of the hyperbolic plane $\mathcal{H}$ under the action of some Fuchsian surface group Γ. Furthermore, a group G of automorphisms of a surface $X = \mathcal{H}/\Gamma$ can be represented as $G = \Lambda/\Gamma$ for another Fuchsian group Λ. Each Fuchsian group Λ is given a signature $\sigma(\Lambda) = (g; m_1, \ldots, m_r)$, where g, m_i are integers verifying $g \geq 0, m_i \geq 2$. The $g = 0$ in signature will be omitted and $m_i = m$ repeated r-times will be written m^r. The signature determines the presentation of Λ:

generators: $x_1, \ldots, x_r, a_1, b_1, \ldots, a_g, b_g$,
relations: $x_1^{m_1} = \ldots = x_r^{m_r} = x_1 \ldots x_r [a_1, b_1] \ldots [a_g, b_g] = 1$.

Such set of generators is called the *canonical set of generators* and often, by abuse of language, the set of *canonical generators*. Geometrically x_i are elliptic elements which correspond to hyperbolic rotations and the remaining generators are hyperbolic translations. The integers $m_1, m_2, \ldots, m_r$ are called the *periods* of Λ and g is the genus of the orbit space $\mathcal{H}/\Lambda$. Fuchsian groups with signatures $(g; -)$ are called *surface groups* and they are characterized among Fuchsian groups as these ones which are torsion free.

The group Λ has associated to it a fundamental region whose area $\mu(\Lambda)$, called the *area of the group*, is:

$$\mu(\Lambda) = 2\pi \left(2g - 2 + \sum_{i=1}^{r} (1 - 1/m_i) \right). \tag{2.1}$$

If Γ is a subgroup of finite index in Λ, then we have the *Riemann-Hurwitz formula* which says that

$$[\Lambda : \Gamma] = \frac{\mu(\Gamma)}{\mu(\Lambda)}. \tag{2.2}$$

The number of fixed points of an automorphism of X can be calculated by the following theorem of Macbeath [19].

Theorem 2.1. *Let $X = H/\Gamma$ be a Riemann surface with the automorphism group $G = \Lambda/\Gamma$ and let $x_1, \dots, x_r$ be elliptic canonical generators of Λ with periods $m_1, \dots, m_r$ respectively. Let $\theta : \Lambda \to G$ be the canonical epimorphism and for $1 \neq g \in G$ let $\varepsilon_i(g)$ be 1 or 0 according whether g is or is not conjugate to a power of $\theta(x_i)$. Then the number $F(g)$ of points of X fixed by g is given by the formula*

$$F(g) = |N_G(\langle g \rangle)| \sum_{i=1}^{r} \varepsilon_i(g)/m_i. \tag{2.3}$$

Let G be a finite group acting on a surface X of genus $g > 1$ such that the canonical projection $X \to X/G$ is ramified at r points with multiplicities $m_1, \dots, m_r$ and s is the genus of X/G. Then a $(2s+r)$-tuple $(\tilde{a}_1, \dots, \tilde{a}_s, \tilde{b}_1, \dots, \tilde{b}_s, \tilde{x}_1, \dots, \tilde{x}_r)$ of generators of G such that $\tilde{x}_i$ has order m_i for $i = 1, \dots, r$, $\tilde{x}_1 \dots \tilde{x}_r \prod_{i=1}^{s} [\tilde{a}_i, \tilde{b}_i] = 1$ and $2g-2 = |G|(2s-2+\sum_{i=1}^{r}(1-1/m_i))$ is called a *generating $(s; m_1, \dots, m_r)$-vector.*

For every generating $(s : m_1, \dots, m_r)$-vector of G, there exists a Fuchsian group Λ with the signature $(s; m_1, \dots, m_r)$ and an epimorphism $\theta : \Lambda \to G$ defined by the assignment $\theta(a_i) = \tilde{a}_i, \theta(b_i) = \tilde{b}_i$ and $\theta(x_j) = \tilde{x}_j$. The kernel Γ of θ is a surface Fuchsian group of orbit genus g and G acts as an automorphism group on a Riemann surface $X = \mathcal{H}/\Gamma$. If an involution ρ appears in generating vector as an image of k consecutive elliptic generators of Λ, then we shall write $\rho^{[k]}$ instead of $\rho, \overset{k}{\dots}, \rho$. There is a $1-1$ correspondence between the set of generating vectors of G and the set of epimorphisms $\theta : \Lambda \to G$ with torsion free kernels. Two epimorphisms $\theta : \Lambda \to G$ and $\theta' : \Lambda' \to G'$ define topologically equivalent actions if $\varphi\theta = \theta'\psi$ for some isomorphisms $\varphi : G \to G'$ and $\psi : \Lambda \to \Lambda'$ [3].

3. p-HYPERELLIPTIC RIEMANN SURFACE WITH (q, n)-GONAL AUTOMORPHISM

In this section we study Riemann surfaces of genera $g > 1$ which are p-hyperelliptic and cyclic (q, n)-gonal simultaneously for a prime $n > 2$ and a natural q. If $g > 4p+1$, then its (q, n)-gonal automorphism and p-hyperelliptic involution commute. The first theorem gives necessary and sufficient conditions on p and g for the existence of such surface.

Theorem 3.1. *There exists a p-hyperelliptic Riemann surface of genus $g \geq 2$ admitting (q, n)-gonal automorphism commuting with a p-hyperelliptic involution if and only if $p = n\gamma + b(n-1)/2$ and $g = nq + a(n-1)/2$ for some integers γ, b, a such that*

$$b = -2 \text{ or } b \geq 0, \quad b \leq a \leq 2(b+1), \quad 0 \leq \gamma \leq (q+1)/2. \tag{3.1}$$

Furthermore, the (q, n)-gonal automorphism admits $a+2$ fixed points.

Proof. Assume that a Riemann surface $X = \mathcal{H}/\Gamma$ admits p-hyperelliptic involution ρ and (q,n)-gonal automorphism δ. The groups $\langle\delta\rangle$ and $\langle\rho\rangle$ can be identified with Γ_δ/Γ and Γ_ρ/Γ, where Γ_δ and Γ_ρ are Fuchsian groups containing Γ as a normal subgroup of index n and 2, respectively. By the Riemann-Hurwitz formula they have signatures

$$\sigma(\Gamma_\delta) = (q; n \overset{r}{\ldots}, n) \text{ and } \sigma(\Gamma_\rho) = (p; 2, \overset{s}{\ldots}, 2), \tag{3.2}$$

where $s = 2g + 2 - 4p$ and $r = 2 + (2g - 2nq)/(n-1)$. Thus $g = nq + a(n-1)/2$ for $a = r - 2$. If ρ and δ commute then they generate the group Z_{2n} which can be represented by Λ/Γ for a Fuchsian group Λ with the signature

$$(\gamma; 2, \overset{k_1}{\ldots}, 2, n, \overset{k_2}{\ldots}, n, 2n, \overset{k_3}{\ldots}, 2n). \tag{3.3}$$

By the Riemann-Hurwith formula

$$2g - 2 = 4n\gamma - 4n + nk_1 + 2k_2(n-1) + k_3(2n-1) \tag{3.4}$$

and according to Theorem 2.1

$$nk_1 = s - k_3, \quad 2k_2 = r - k_3. \tag{3.5}$$

By substituting the last equalities to (3.4), we obtain $p = n\gamma + b(n-1)/2$, for an integer b such that $a = 2b + 2 - k_3$. Thus

$$k_1 = 2q + a - 4\gamma - 2b, \quad k_2 = a - b, \quad k_3 = 2 + 2b - a \tag{3.6}$$

are nonnegative integers if and only if the inequalities (3.1) are satisfied.

Conversely, assume that $g = nq + a(n-1)/2$ and $p = n\gamma + b(n-1)/2$ for some integers a, b and γ satisfying the inequalities (3.1). Then there exists a Fuchsian group Λ with the signature (3.3). Let $\theta : \Lambda \to \langle\rho\rangle \oplus \langle\delta\rangle$ be an epimorphism which maps all hyperbolic generators of Λ onto $\rho\delta$, the first k_1 of elliptic generators onto ρ and the remaining in the following way :

$$\underbrace{\delta\ldots\delta}_{(k_2+1)/2}\ \underbrace{\delta^{-1}\ldots\delta^{-1}}_{(k_2-3)/2}\ \delta^{-2}\ \underbrace{\rho\delta\ldots\rho\delta}_{(k_3+1)/2}\ \underbrace{\rho\delta^{-1}\ldots\rho\delta^{-1}}_{(k_3-3)/2}\ \rho\delta^{-2} \text{ if } k_2 \equiv 1\ (2) \text{ and } k_3 \equiv 1\ (2),$$

$$\underbrace{\delta\ldots\delta}_{(k_2+1)/2}\ \underbrace{\delta^{-1}\ldots\delta^{-1}}_{(k_2-3)/2}\ \delta^{-2}\ \underbrace{\rho\delta\ldots\rho\delta}_{k_3/2}\ \underbrace{\rho\delta^{-1}\ldots\rho\delta^{-1}}_{k_3/2} \text{ if } k_2 \equiv 1\ (2) \text{ and } k_3 \equiv 0\ (2),$$

$$\underbrace{\delta\ldots\delta}_{k_2/2}\ \underbrace{\delta^{-1}\ldots\delta^{-1}}_{k_2/2}\ \underbrace{\rho\delta\ldots\rho\delta}_{(k_3+1)/2}\ \underbrace{\rho\delta^{-1}\ldots\rho\delta^{-1}}_{(k_3-3)/2}\ \rho\delta^{-2} \text{ if } k_2 \equiv 0\ (2) \text{ and } k_3 \equiv 1\ (2),$$

$$\underbrace{\delta\ldots\delta}_{k_2/2}\ \underbrace{\delta^{-1}\ldots\delta^{-1}}_{k_2/2}\ \underbrace{\rho\delta\ldots\rho\delta}_{k_3/2}\ \underbrace{\rho\delta^{-1}\ldots\rho\delta^{-1}}_{k_3/2} \text{ if } k_2 \equiv 0\ (2) \text{ and } k_3 \equiv 0\ (2).$$

Then the kernel of θ is a surface Fuchsian group Γ of genus g while $\theta^{-1}(\rho)$ and $\theta^{-1}(\delta)$ are Fuchsian groups with the signatures (3.2). Thus $\mathcal{H}/\Gamma$ is a p-hyperelliptic Riemann surface admitting (q,n)-gonal automorphism. It is easy to notice that for $k_2 < 3$ or $k_3 < 3$, such an epimorphism does not exist if and only if $k_2 + k_3 + \gamma = 0$ or $k_2 + k_3 = 1$. The first equality is never satisfied since if $k_2 + k_3 = 0$ then $b = -2$ and $p = n(\gamma - 1) + 1$ what requires $\gamma \geq 1$. The second one occurs for $b = -1$ and therefore this value of b is rejected. □

Corollary 3.2. *Let X be a p-hyperelliptic Riemann surface of genus $g > 4p+1$. Then for any prime $n \geq 3$,*

(i) *X can be realized as n-sheeted covering of the Riemann sphere if and only if $p = 0$ and $g = n-1$ or $g = (n-1)/2$ and its n-gonality automorphism admits 4 or 3 fixed points, respectively.*

(ii) *X can be realized as n-sheeted covering of an elliptic curve if and only if $p = 0$ and $g \in \{2n-1, (3n-1)/2, n\}$ or $p = (n-1)/2$ and $g \in \{3n-2, (5n-3)/2\}$ and its $(1, n)$-gonal automorphism admits $4, 3, 2$ or $6, 5$ fixed points, respectively.*

Corollary 3.3. *Let $X = \mathcal{H}/\Gamma$ be a Riemann surface of genus $g \geq 2$ which admits p-hyperelliptic involution ρ and (q, n)-gonal automorphism δ for $p < n$. If δ and ρ commute then $p = b(n-1)/2$, $g = nq + a(n-1)/2$ for integers a, b in range $0 \leq b \leq 2$ and $b \leq a \leq 2b+2$ and a Fuchsian group Λ such that $\langle \delta, \rho \rangle = \Lambda/\Gamma$ has a signature $(0; 2, \overset{2q+a-2b}{\ldots}, 2, n, \overset{a-b}{\ldots}, n, 2n, \overset{2b+2-a}{\ldots}, 2n)$. Furthermore, δ admits $a+2 \leq 8$ fixed points.*

Theorem 3.4. *All group actions on a p-hyperelliptic and cyclic n-gonal Riemann surface are given in Table 1, up to topological conjugacy; four of them correspond to the full automorphism groups: $2.b, 3.a, 3.b$ and $5.c$.*

Proof. Let $X = \mathcal{H}/\Gamma$ be a p-hyperelliptic Riemann surface of genus $g \geq 2$ admitting a n-gonality automorphism δ. Then by Corollary 3.2, X is hyperelliptic, δ admits 4 or 3 fixed points and its order is one of two possible prime orders greater than g, namely $n = g+1$ or $n = 2g+1$, respectively. The automorphism groups of hyperelliptic Riemann surfaces are given in [12] and we need to chose those which admit an automorphism satisfying the above conditions. The action of finite group G on X is determined by the signature of a Fuchsian group Λ and an epimorphism $\theta : \Lambda \to G$ with kernel Γ. Let $x_1, \ldots, x_r$ be all elliptic generators of Λ. An element of Λ has a fixed point in $\mathcal{H}$ if and only if it has a finite order and it is conjugate to some power of precisely one of elliptic generators x_i. Consequently an element of G has a fixed point in X if and only if it is conjugate to some power of the image of x_i via homomorphism θ. Since θ preserves orders, it follows that the order n of the n-gonality automorphism divides one of periods m_i in the signature of Λ. First we chose all signatures corresponding to group actions on a hyperelliptic Riemann surface of genus g for which $g+1$ or $2g+1$ divides one of its periods. The authors of [12] denoted by t the number of periods 2 in the signature of Λ which correspond to elliptic generators mapped by θ on the hyperelliptc involution and expressed t in terms of the genus g and the the number $N = |G|/2$. Let us consider for example $\sigma(\Lambda) = (2, \overset{t}{\ldots}, 2, 2, 3, 3)$ with $t = (g+1)/6$. The number 3 is the only prime integer greater that 2 which divides a period of Λ. Thus δ has order 3 and so $g = 2$. However t is not integer for $t = 2$ and therefore this signature is not suitable. In the similar way we reject the remaining signatures except:

$2.a: \quad \sigma(\Lambda) = (2, \overset{t}{\ldots}, 2, N, N), \quad t = (2g+2)/N,$

$2.b: \quad \sigma(\Lambda) = (2, \overset{t}{\ldots}, 2, N, 2N), \quad t = (2g+1)/N,$

$$\begin{array}{lll} 3.a: & \sigma(\Lambda) = (2, \overset{t}{\ldots}, 2, 2, 2, N/2), & t = (2g+2)/N, \\ 3.b: & \sigma(\Lambda) = (2, \overset{t}{\ldots}, 2, 2, 4, N/2), & t = (2g+2)/N - 1/2, \\ 3.c: & \sigma(\Lambda) = (2, \overset{t}{\ldots}, 2, 4, 4, N/2), & t = (2g+2)/N - 1, \\ 4.d: & \sigma(\Lambda) = (2, \overset{t}{\ldots}, 2, 4, 3, 3), & t = (g-2)/6, g = 2, \\ 5.c: & \sigma(\Lambda) = (2, \overset{t}{\ldots}, 2, 2, 3, 8), & t = (g-2)/12, g = 2. \end{array}$$

In the case $2.a$, $G = \langle z : z^2 \rangle \oplus \langle x : x^N \rangle$ and z is the hyperelliptic involution. The order n of δ divides a period of Λ if and only if $n = g+1$ and N has one of values $2g+2$ or $g+1$. Thus $\langle \delta \rangle = \langle x^2 \rangle$ or $\langle x \rangle$, respectively and we shall denote these two possibilities by $2.a$ and $2.a'$ in Table 1. With the help of Macbeath's theorem we check that in both cases δ has 4 fixed points as required. Using the pair of automorphisms $(\mathrm{id}_\Lambda, \varphi)$, where $\varphi(x) = xz$ and $\varphi(z) = z$ if necessary, we can show that any generating vector is equivalent to $v = (z, \overset{t}{\ldots}, z, xz^t, x^{-1})$. A similar consideration of the all signatures listed above provides the remaining results in Table 1.

Table 1. Actions on a p-hyperelliptic cyclic n-gonal Riemann surface

	$\sigma(\Lambda)$	$G = \Lambda/\Gamma$ of oder $2N$	N	gen. vector	δ
$2.a$	$[2, N, N]$	$\langle z : z^2 \rangle \oplus \langle x : x^N \rangle$	$2g+2$	(z, zx, x^{-1})	x^2
$2.a'$	$[2, 2, N, N]$	$\langle z : z^2 \rangle \oplus \langle x : x^N \rangle$	$g+1$	(z, z, x, x^{-1})	x
$2.b$	$[2, N, 2N]$	$\langle x : x^{2N} \rangle$	$2g+1$	(x^N, x^2, x^{N-2})	x^2
$3.a$	$[2, 2, 2, N]$	$\langle z : z^2 \rangle \oplus \langle x, y : x^2, y^2, (xy)^N \rangle$	$g+1$	$(z, zx, y, (xy)^{-1})$	xy
$3.b$	$[2, 4, N/2]$	$\langle x, y : x^4, y^{N/2}, (xy)^2, (x^{-1}y)^2 \rangle$	$4g+4$	$((xy)^{-1}, x, y)$	y^2
$3.c$	$[4, 4, N]$	$\langle x, y : x^4, x^2y^2, (xy)^N \rangle$	$g+1$	$(x, y, (xy)^{-1})$	xy
$4.d$	$[4, 3, 3]$	$\langle x, y : x^4, y^3, (xy)^3, yx^2y^{-1}x^2 \rangle$	12	$(x, y, (xy)^{-1})$	y
$5.c$	$[2, 3, 8]$	$\langle x, y : x^2, y^3, (xy)^4(yx)^4, (xy)^8 \rangle$	24	$(x, y, (xy)^{-1})$	xyx

If the signature of Λ does not appear in the first column of the Tables 1.5.1 or 1.5.2 in [25] then Λ can be chosen to be maximal [25] and so G can be assumed to be the full group of automorphisms of X. In the other case Λ is always contained in a Fuchsian group Λ' and the signature of of such a group is given in the second column of the corresponding row, what we shall denote by $\sigma(\Lambda) \subset \sigma(\Lambda')$. By inspecting the signatures from Table 1 we obtain: $[2, 2g+2, 2g+2] \subset [2, 4, 2g+2]$, $[2, 2, g+1, g+1] \subset [2, 2, 2, g+1]$, $[4, 4, g+1] \subset [2, 4, 2g+2]$, $[4, 3, 3] \subset [2, 3, 8]$ and $[2, N, 2N] \subset 2, 3, 2N]$. In each of these cases except the last one, there exists a group G' acting on a hyperelliptic Riemann surface of genus g, group embeddings $i : \Lambda \hookrightarrow \Lambda'$, $j : G \hookrightarrow G'$ and an epimorphism $\theta' : \Lambda' \to G'$ such that $[\Lambda' : \Lambda] = [G' : G]$ and $\theta' i = j\theta$. In the last case the genus of a surface on which G' acts is different from g. Consequently G is the full automorphism group of a hyperelliptic Riemann surface only in cases $2.b$, $3.a$, $3.b$ and $5.c$. $\square$

Using Corollary 3.2, Macbeath's theorem and group actions on hyperelliptic, elliptic-hyperelliptic and 2-hyperelliptic Riemann surfaces given, up to topological conjugacy, in [12, 20] and [21], we obtain the next theorems. Their proofs are similar to the previous one and so we omit them.

Theorem 3.5. *A p-hyperelliptic Riemann surface of genus $g > 4p+1$ can be realized as cyclic 3-sheeted covering of an elliptic curve if and only if $p = 0$ and $g = 3, 4, 5$ or $p = 1$ and $g = 6, 7$ while the topologically non-equivalent group actions on such surfaces are listed in Table 2.*

Theorem 3.6. *A p-hyperelliptic Riemann surface of genus $g > 4p+1$ can be realized as cyclic 5-sheeted covering of an elliptic curve if and only if $p = 0$ and $g = 5, 7, 9$ or $p = 2$ and $g = 11, 13$ while the topologically non-equivalent group actions on such surfaces are listed in Table 3.*

Theorem 3.7. *For any prime $n > 5$, a hyperelliptic $(1, n)$-gonal Riemann surface has genus $2n-1, (3n-1)/2$ or n and the finite group actions on such surfaces are given in Table 4.*

Table 2. Actions on a p-hyperelliptic cyclic $(1, 3)$-gonal Riemann surface

g	$\sigma(\Lambda)$	$G = \Lambda/\Gamma$	gen. vector	ρ	δ
3	$[2^2, 6^2]$	$\langle x : x^6\rangle$	$(\rho^{[2]}, x, x^{-1})$	x^3	x^2
	$[2, 6^2]$	$\langle z : z^2\rangle \oplus \langle x, y : x^2, y^3, (xy)^3\rangle$	$(x, \delta\rho, (x\delta)^{-1}\rho)$	z	y
	$[2, 6, 4]$	$\langle z : z^2\rangle \oplus \langle x, y : x^2, y^3, (xy)^4\rangle$	$(x, \delta\rho, (x\delta)^{-1}\rho)$	z	y
	$[2, 12^2]$	$\langle x : x^{12}\rangle$	(ρ, x^7, x^{-1})	x^6	x^4
	$[2^3, 6]$	$\langle x, y : x^2, y^2, (xy)^6\rangle$	$(\rho, \rho x, y, \rho\delta$	$(xy)^3$	$(xy)^2$
	$[4^2, 6]$	$\langle x, y : x^2y^3, y^6, x^{-1}yxy\rangle$	$(x, (yx)^{-1}, y)$	x^2	y^2
4	$[4, 3, 6]$	$\langle x, y : x^4, y^3, (xy)^3, yx^2y^{-1}x^2\rangle$	$(x, \delta, (x\delta)^{-1})$	x^2	y
	$[2^3, 3, 6]$	$\langle x : x^6\rangle$	$(\rho^{[3]}, \delta, x)$	x^3	x^2
	$[2, 9, 18]$	$\langle x : x^{18}\rangle$	(ρ, x^2, x^7)	x^9	x^6
5	$[2^2, 3^2]$	$\langle z : z^2\rangle \oplus \langle x, y : x^2, y^3, (xy)^3\rangle$	$(\rho, \rho x, \delta, (x\delta)^{-1})$	z	y
	$[4, 3, 4]$	$\langle x, y : x^4, y^3, yx^2y^{-1}x^2, (xy)^4\rangle$	$(x, \delta, (x\delta)^{-1})$	x^2	y
	$[2^4, 3^2]$	$\langle z : z^2\rangle \oplus \langle x : x^3\rangle$	$(\rho^{[4]}, \delta, \delta^{-1})$	z	x
	$[2^2, 6^2]$	$\langle z : z^2\rangle \oplus \langle x : x^6\rangle$	$(\rho^{[2]}, x, x^{-1})$	z	x^2
	$[2, 12^2]$	$\langle z : z^2\rangle \oplus \langle x : x^{12}\rangle$	$(\rho, \rho x^{-1}, x)$	z	x^4
	$[2^4, 3]$	$\langle z : z^2\rangle \oplus \langle x, y : x^2, y^2, (xy)^3\rangle$	$(\rho^{[2]}, x, y, \delta^{-1})$	z	xy
	$[2, 4^2, 3]$	$\langle x, y : x^4, x^2y^2, (xy)^3\rangle$	$(\rho, x^3, y, \delta^{-1})$	x^2	xy
	$[4^2, 6]$	$\langle x, x : x^4, x^2y^2, (xy)^6\rangle$	$(x, y, (xy)^{-1})$	x^2	$(xy)^2$
6	$[2^3, 3^2, 6]$	$\langle z : z^2\rangle \oplus \langle c : c^3\rangle$	$(\rho^{[3]}, \delta, \delta^{-2}, \rho\delta)$	z	c
	$[2, 4, 3, 12]$	$\langle c : c^{12}\rangle$	$(\rho, c^3, \delta, \rho\delta)$	c^6	c^4
7	$[4, 3, 6]$	$\langle x, y, c, z : z^2, c^6, y^2z, x^2z, [x, y]z,$ $cyc^{-1}y^{-1}x, cxc^{-1}y^{-1}z, [z, c]\rangle$	(c^3x, c^2y, c)	z	c^4
	$[2^3, 3, 6]$	$\langle z : z^2\rangle \oplus \langle c : c^6\rangle$	$(\rho^{[2]}, c^3, \delta, c)$	z	c^2
	$[2^4, 3^3]$	$\langle z : z^2\rangle \oplus \langle c : c^3\rangle$	$(\rho^{[4]}, \delta, \delta^{-2}, \delta)$	z	c
	$[2, 3^2, 6]$	$\langle z : z^2\rangle \oplus \langle y : y^3\rangle \oplus \langle c : c^3\rangle$	$(\rho\delta, \delta y^2, y\delta\rho)$	z	c
	$[2, 3, 12]$	$\langle x, y, c : c^{12}, c^6y^{-6}, x^2y^2, xyx^{-1}y^5,$ $cxc^{-1}y^{-1}, cyc^{-1}y^{-1}x\rangle$	(c^3x, c^2y, c)	c^6	c^4
	$[3^2, 6]$	$\langle x, y, c, z : z^2, c^3, y^6z, [x, y]z, x^2y^2,$ $cxc^{-1}y^{-1}x, cyc^{-1}x, [c, z], [x, z]\rangle$	$(\delta, \delta x, x^{-1}\delta)$	z	c

Table 3. Actions on p-hyperelliptic cyclic $(1,5)$-gonal Riemann surfaces

g	$\sigma(\Lambda)$	$G=\Lambda/\Gamma$	gen. vector	ρ	δ
5	$[2^2,10^2]$	$\langle x:x^{10}\rangle$	$(\rho^{[2]},x,x^{-1})$	x^5	x^2
	$[2,20^2]$	$\langle x:x^{20}\rangle$	$(\rho,\rho x,x^{-1})$	x^{10}	x^4
	$[2^3,10]$	$\langle x,y:x^2,y^2,(xy)^{10}\rangle$	$(\rho,\rho x,y,(xy)^{-1})$	$(xy)^5$	$(xy)^2$
	$[4^2,10]$	$\langle x,y:x^2y^5,y^{10},x^{-1}yxy\rangle$	$(x,(yx)^{-1},y)$	x^2	y^2
	$[2,3,10]$	$\langle z:z^2\rangle\oplus\langle x,y:x^2,y^3,(xy)^5\rangle$	$(\rho x,y,\delta^2\rho)$	z	$(xy)^2$
7	$[2^3,5,10]$	$\langle x:x^{10}\rangle$,	$(\rho^{[3]},\delta,(\rho\delta)^{-1})$	x^5	x^2
	$[2,15,30]$	$\langle x:x^{30}\rangle$,	$(\rho,x^2,x\delta^2)$	x^{15}	x^6
9	$[2^4,5^2]$	$\langle z:z^2\rangle\oplus\langle x:x^5\rangle$	$(\rho^{[4]},\delta,\delta^{-1})$	z	x
	$[2^2,10^2]$	$\langle z:z^2\rangle\oplus\langle x:x^{10}\rangle$	$(\rho^{[2]},x,x^{-1})$	z	x^2
	$[2,20^2]$	$\langle z:z^2\rangle\oplus\langle x:x^{20}\rangle$	$(\rho,\rho x^{-1},x)$	z	x^4
	$[2^4,5]$	$\langle z:z^2\rangle\oplus\langle x,y:x^2,y^2,(xy)^5\rangle$	$(\rho^{[2]},x,y,\delta^{-1})$	z	xy
	$[2,4^2,5]$	$\langle x,y:x^4,x^2y^2,(xy)^5\rangle$	$(\rho,\rho x,y,\delta^{-1})$	x^2	xy
	$[4^2,10]$	$\langle x,x:x^4,x^2y^2,(xy)^{10}\rangle$	$(x,y,(xy)^{-1})$	x^2	$(xy)^2$
	$[2,6,5]$	$\langle z:z^2\rangle\oplus\langle x,y:x^2,y^3,(xy)^5\rangle$	$(\rho x,y\rho,\delta^{-1})$	z	xy
11	$[10,5^2,2^3]$	$\langle z:z^2\rangle\oplus\langle x:x^5\rangle$	$(\delta\rho,\delta,\delta^3,\rho^{[3]})$	z	x
	$[4,5,20,2]$	$\langle x:x^{20}\rangle$	$(\delta x,\delta,x,\rho)$	x^{10}	x^4
13	$[5^3,2^4]$	$\langle z:z^2\rangle\oplus\langle x:x^5\rangle$	$(\delta,\delta,\delta^3,\rho^{[4]}$	z	x
	$[2,5,10,2^2]$	$\langle z:z^2\rangle\oplus\langle x:x^{10}\rangle$	$(\delta^2x,\delta^2,x,\rho^{[2]})$	z	x^2

Table 4. Actions on a hyperelliptic cyclic $(1,n)$-gonal Riemann surface for $n>5$

g	$\sigma(\Lambda)$	$G=\Lambda/\Gamma$	gen. vector	ρ	δ
$2n-1$	$[2^4,n^2]$	$\langle z:z^2\rangle\oplus\langle x:x^n\rangle$	$(\rho^{[4]},\delta,\delta^{-1})$	z	x
	$[2^2,(2n)^2]$	$\langle z:z^2\rangle\oplus\langle x:x^{2n}\rangle$	$(\rho^{[2]},x,x^{-1})$	z	x^2
	$[2,(4n)^2]$	$\langle z:z^2\rangle\oplus\langle x:x^{4n}\rangle$	$(\rho,\rho x^{-1},x)$	z	x^4
	$[2^4,n]$	$\langle z:z^2\rangle\oplus\langle x,y:x^2,y^2,(xy)^n\rangle$	$(\rho^{[2]},x,y,\delta^{-1})$	z	xy
	$[2,4^2,n]$	$\langle x,y:x^4,x^2y^2,(xy)^n\rangle$	$(\rho,\rho x,y,\delta^{-1})$	x^2	xy
	$[4^2,2n]$	$\langle x,x:x^4,x^2y^2,(xy)^{2n}\rangle$	$(x,y,(xy)^{-1})$	x^2	$(xy)^2$
$\frac{3n-1}{2}$	$[2^3,n,2n]$	$\langle x:x^{2n}\rangle$	$(\rho^{[3]},\delta,x^n\delta^{-1})$	x^n	x^2
	$[2,3n,6n]$	$\langle x:x^{6n}\rangle$	$(\rho,x^2,\rho x^{-2})$	x^{3n}	x^6
n	$[2^2,(2n)^2]$	$\langle x:x^{2n}\rangle$	$(\rho^{[2]},x,x^{-1})$	x^n	x^2
	$[2,(4n)^2]$	$\langle x:x^{4n}\rangle$	$(\rho,\rho x,x^{-1})$	x^{2n}	x^4
	$[2^3,2n]$	$\langle x,y:x^2,y^2,(xy)^{2n}\rangle$	$(\rho,x,y,(xy)^{n-1})$	$(xy)^n$	$(xy)^2$
	$[4^2,2n]$	$\langle x,y:x^2y^n,y^{2n},x^{-1}yxy\rangle$	$(x,(yx)^{-1},y)$	x^2	y^2

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