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ON CHROMATIC EQUIVALENCE OF A PAIR OF K_4-HOMEOMORPHS

Abstract. Let $P(G,\lambda)$ be the chromatic polynomial of a graph G. Two graphs G and H are said to be chromatically euqivalent, denoted $G \sim H$, if $P(G,\lambda) = P(H,\lambda)$. We write $[G] = \{H|H \sim G\}$. If $[G] = \{G\}$, then G is said to be chromatically unique. In this paper, we discuss a chromatically equivalent pair of graphs in one family of K_4-homeomorphs, $K_4(1,2,8,d,e,f)$. The obtained result can be extended in the study of chromatic equivalence classes of $K_4(1,2,8,d,e,f)$ and chromatic uniqueness of K_4-homeomorphs with girth 11.

Keywords: chromatic polynomial, chromatic equivalence, K_4-homeomorphs.

Mathematics Subject Classification: 05C15.

1. INTRODUCTION

All graphs considered here are simple graphs. For such a graph G, let $P(G,\lambda)$ (or simply $P(G)$) denote the chromatic polynomial of G. Two graphs G and H are chromatically equivalent (or simply χ-equivalent), denoted by $G \sim H$, if $P(G,\lambda) = P(H,\lambda)$ (or simply $P(G) = P(H)$. A graph G is chromatically unique (or simply χ-unique) if for any graph H such that $H \sim G$, we have $H \cong G$, i.e, H is isomorphic to G. A K_4-homeomorph is a subdivision of the complete graph K_4. Such a homeomorph is denoted by $K_4(a,b,c,d,e,f)$ if the six edges of K_4 are replaced by the six paths of length a, b, c, d, e, f, respectively, as shown in Figure 1. So far, the chromaticity of K_4-homeomorphs with girth g, where $3 \leq g \leq 9$ has been studied by many authors (see [5, 9–11, 18]). In 2004, Peng in [9] published her work on the chromaticity of K_4-homeomorphs with girth six by considering her result on the chromatic equivalence pair $K_4(1,2,3,d,e,f)$ and $K_4(1,2,3,d',e',f')$. Dong et. al in [6] summarized the above result. In 2008, Peng [11] investigated the chromatic uniqueness of $K_4(1,3,3,d,e,f)$ with exactly one path of length one and with girth seven. She accomplished this, first by establishing the chromatic equivalence pair of $K_4(1,3,3,d,e,f)$ and $K_4(1,3,3,d',e',f')$ in [12]. She then solved the chromatic equivalence of such families of graphs (see [12–14]) and finally, in [11], she provided the

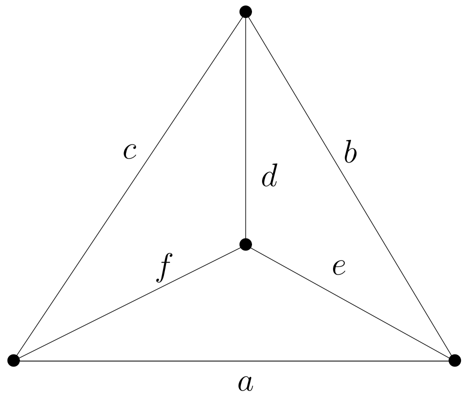

Fig. 1. $K_4(a, b, c, d, e, f)$

necessary and sufficient condition for this type of K_4-homeomorph to be chromatically unique. S. Catada-Ghimire et al. in [1] investigated the chromaticity of one family of K_4-homeomorph with girth 10. For the purpose of completing their on going research on K_4-homeomorphs with the said girth, they published their results on three chromatic equivalence pairs of K_4-homeomorphs in [2, 3] and [4] which are summarised as follows:

Let $G = K_4(1, b, c, d, e, f)$ and $H = K_4(1, b, c, d', e', f')$ be non-isomorphic but chromatically equivalent. Then $\{G, H\}$ is one of the following pairs:

when $b = b' = 2$ and $c = c' = 7$

$$\{K_4(1, 2, 7, i, i + 8, i + 1), K_4(1, 2, 7, i + 2, i, i + 7)\},$$
$$\{K_4(1, 2, 7, i, i + 1, i + 8), K_4(1, 2, 7, i + 7, i, i + 2)\},$$
$$\{K_4(1, 2, 7, i, i + 1, i + 3), K_4(1, 2, 7, i + 2, i + 2, i)\},$$

when $b = b' = 3$ and $c = c' = 6$

$$\{K_4(1, 3, 6, i, i + 1, i + 4), K_4(1, 3, 6, i + 2, i + 3, i)\},$$
$$\{K_4(1, 3, 6, i, i + 7, i + 1), K_4(1, 3, 6, i + 2, i, i + 6)\},$$

when $b = b' = 4$ and $c = c' = 5$

$$\{K_4(1, 4, 5, i, i + 6, i + 1), K_4(1, 4, 5, i + 2, i, i + 5)\},$$
$$\{K_4(1, 4, 5, i, i + 1, i + 5), K_4(1, 4, 5, i + 2, i + 4, i)\}.$$

Our main aim is to provide a result which can be extended in the study of the chromatic equivalence of $K_4(1, 2, 8, d, e, f)$ (as shown in Fig. 2). Such results are an indispensable tool in the study of the chromatic uniqueness of K_4-homeomorphs with girth 11.

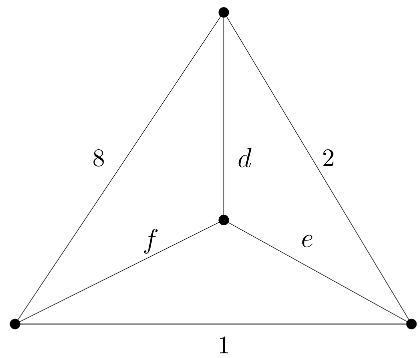

Fig. 2. $K_4(1, 2, 8, d, e, f)$

2. PRELIMINARY RESULT

In this section, we give the following known result used in the sequel.

Lemma 2.1. *Assume that G and H are χ-equivalent. Then:*

(1) $|V(G)| = |V(H)|$, $|E(G)| = |E(H)|$ (see [7]).
(2) *G and H have the same girth and same number of cycles with length equal to their girth* (see [15]).
(3) *If G is a K_4-homeomorph, then H must itself be a K_4-homeomorph* (see [16]).
(4) *Let $G = K_4(a, b, c, d, e, f)$ and $H = K_4(a', b', c', d', e', f')$, then:*
  (i) *min $\{a, b, c, d, e, f\}$ = min $\{a', b', c', d', e', f'\}$ and the number of times that this minimum occurs in the list $\{a, b, c, d, e, f\}$ is equal to the number of times that this minimum occurs in the list $\{a', b', c', d', e', f'\}$* (see [17]);
  (ii) *if $\{a, b, c, d, e, f\} = \{a', b', c', d', e', f'\}$ as multisets, then $H \cong G$* (see [18]).

3. MAIN RESULT

Lemma 3.1. *Let $G \cong K_4(1, 2, 8, d, e, f)$ and $H \cong K_4(1, 2, 8, d', e', f')$, then:*

(1) $P(G) = (-1)^{x-1}[s/(s-1)^2][-s^{x-1} - s^9 - s^8 - s^3 - s^2 + 2s + 2 + R(G)]$, *where* $R(G) = -s^d - s^e - s^f - s^{e+1} - s^{f+1} + s^{f+2} + s^{e+3} + s^{e+8} + s^{f+9} + s^{d+10} + s^{d+e+f}$, $s = 1 - \lambda$, *x is the number of edges of G.*
(2) *If $P(G) = P(H)$, then $R(G) = R(H)$.*

Proof. (1) Let $s = 1 - \lambda$. From [17], the chromatic polynomial of K_4-homeomorphs $K_4(a, b, c, d, e, f)$ is as follows:
$P(K_4(a, b, c, d, e, f) = (-1)^{x-1}[s/(s-1)^2][(s^2 + 3s + 2) - (s + 1)(s^a + s^b + s^c + s^d + {} + s^e + s^f) + (s^{a+d} + s^{b+f} + s^{c+e} + s^{a+b+e} + s^{b+d+c} + s^{a+c+f} + s^{d+e+f} - s^{x-1})]$.

So when $a = 1, b = 2$ and $c = 8$, we have
$P(K_4(1, 2, 8, d, e, f) = (-1)^{x-1}[s/(s-1)^2][(s^2+3s+2)-(s+1)(s+s^2+s^8+s^d+s^e+s^f)+$

$+(s^{d+1}+s^{f+2}+s^{e+3}+s^{e+8}+s^{d+10}+s^{f+9}+s^{d+e+f}-s^{x-1})]=$
$=(-1)^{x-1}[s/(s-1)^2][-s^{x-1}-s^9-s^8-s^3-s^2+2s+2-s^d-s^e-s^f-s^{e+1}-s^{f+1}+$
$+s^{f+2}+s^{e+3}+s^{e+8}+s^{f+9}+s^{d+10}+s^{d+e+f}]=$
$=(-1)^{x-1}[s/(s-1)^2][-s^{x-1}-s^9-s^8-s^3-s^2+2s+2+R(G)]$, where
$R(G)=-s^d-s^e-s^f-s^{e+1}-s^{f+1}+s^{f+2}+s^{e+3}+s^{e+7}+s^{f+9}+s^{d+10}+s^{d+e+f}$
as required.

(2) If $P(G)=P(H)$, then we can easily see that $R(G)=R(H)$. □

Theorem 3.2. *Let K_4-homeomorphs $K_4(1,2,8,d,e,f)$ and $K_4(1,2,8,d',e',f')$ be chromatically equivalent, then we have*

$$K_4(1,2,8,i,i+9,i+1)\sim K_4(1,2,8,i+2,i,i+8),$$
$$K_4(1,2,8,i,i+1,i+9)\sim K_4(1,2,8,i+8,i,i+2),$$
$$K_4(1,2,8,i,i+1,i+3)\sim K_4(1,2,8,i+2,i+2,i),$$

where $i\geq 1$.

Proof. Let $G\cong K_4(1,2,8,d,e,f)$ and $H\cong K_4(1,2,8,d',e',f')$. We now solve for the equation $R(G)=R(H)$ to find G and H which are not isomorphic. From Lemma 3.1, we have

$R(G)=-s^d-s^e-s^f-s^{e+1}-s^{f+1}+s^{f+2}+s^{e+3}+s^{e+8}+s^{f+9}+s^{d+10}+s^{d+e+f}$,
$R(H)=-s^{d'}-s^{e'}-s^{f'}-s^{e'+1}-s^{f'+1}+s^{f'+2}+s^{e'+3}+s^{e'+8}+s^{f'+9}+s^{d'+10}+s^{d'+e'+f'}$.

Let the lowest remaining power and the highest remaining power be denoted by l.r.p. and h.r.p., respectively. From Lemma 2.1 (1), $d+e+f=d'+e'+f'$. We obtain the following after simplification: (Note that our assumption in the following steps of the proof is $R_j(G)=R_j(H)$, where $1\leq j\leq 18$.)

$R_1(G)=-s^d-s^e-s^f-s^{e+1}-s^{f+1}+s^{f+2}+s^{e+3}+s^{e+8}+s^{f+9}+s^{d+10}$,
$R_1(H)=-s^{d'}-s^{e'}-s^{f'}-s^{e'+1}-s^{f'+1}+s^{f'+2}+s^{e'+3}+s^{e'+8}+s^{f'+9}+s^{d'+10}$.

Let us consider the h.r.p. in $R_1(G)$ and the h.r.p. in $R_1(H)$. We have max $\{e+8,f+9,d+10\}=$ max $\{e'+8,f'+9,d'+10\}$. Without loss of generality, we will consider only the following six cases.

Case 1. If max $\{e+8,f+9,d+10\}=e+8$ and max $\{e'+8,f'+9,d'+10\}=e'+8$, then $e=e'$. Thus, we can cancel the following pairs of terms in the equations $R_1(G)$ and $R_1(H)$: $-s^e$ with $-s^{e'}$, $-s^{e+1}$ with $-s^{e'+1}$, s^{e+3} with $s^{e'+3}$ and s^{e+8} with $s^{e'+8}$. Therefore, the l.r.p. in $R_1(G)$ is d or f and the l.r.p. in $R_1(H)$ is d' or f'. So, $d=f'$ or $d=d'$ or $f=f'$ or $f=d'$. We have $e=e'$ and $d+e+f=d'+e'+f'$. So, we know that $\{d,e,f\}=\{d',e',f'\}$ as multisets. From Lemma 2.1 (4(ii)), $G\cong H$.

Case 2. If max $\{e+8,f+9,d+10\}=f+9$ and max $\{e'+8,f'+9,d'+10\}=f'+9$, then $f=f'$. We can deal with this case in the same way as case 1, thus, $G\cong H$.

Case 3. If max $\{e+8,f+9,d+10\}=d+10$ and max $\{e'+8,f'+9,d'+10\}=d'+10$, then we can deal with this case in the same way as case 1. So, we have $G\cong H$.

Case 4. If max $\{e+8,f+9,d+10\}=e+8$ and max $\{e'+8,f'+9,d'+10\}=f'+9$, then $e+8=f'+9$, that is

$$f'=e-1 \tag{3.1}$$

from $d+e+f=d'+e'+f'$, we have

$$d+f=d'+e'-1. \tag{3.2}$$

Consider the l.r.p. in $R_1(G)$ and the l.r.p. in $R_1(H)$. From Lemma 2.1(4(i)), min $\{d,e,f\}$ = min $\{d',e',f'\}$. Without loss of generality, let min $\{d,e,f\}=d$. The following subcases need to be considered.

Subcase 4.1. If min $\{d,e,f\}=d$ and min $\{d',e',f'\}=d'$, then $d=d'$. Thus, we can consider this case the same way as case 1. So, $G\cong H$.

Subcase 4.2. If min $\{d,e,f\}=d$ and min $\{d',e',f'\}=e'$, then $d=e'$. From Eq. (3.2), we have $d'=f+1$. Note that $f'=e-1$ (Eq. (3.1)). We can write $R_1(G)$ and $R_1(H)$ as follows:

$R_2(G)=-s^d-s^e-s^f-s^{e+1}-s^{f+1}+s^{f+2}+s^{e+3}+s^{e+8}+s^{f+9}+s^{d+10}$

$R_2(H)=-s^{f+1}-s^d-s^{e-1}-s^{d+1}-s^e+s^{e+1}+s^{d+3}+s^{d+8}+s^{e+8}+s^{f+11}$.

After simplifying $R_2(G)$ and $R_2(G)$, we have

$R_3(G)=-s^f-s^{e+1}+s^{f+2}+s^{e+3}+s^{f+9}+s^{d+10}$

$R_3(H)=-s^{e-1}-s^{d+1}+s^{e+1}+s^{d+3}+s^{d+8}+s^{f+11}$.

Consider the term $-s^{d+1}$ in $R_3(H)$. Since the min $d,e,f=d$, $-s^{d+1}$ cannot be cancelled by any of the positive terms in $R_3(H)$. Thus, $-s^{d+1}$ must be equal to $-s^f$ or $-s^{e+1}$ in $R_3(G)$. Note that max $e+8,f+9,d+10=e+8$, so $e+8\geq d+10$, that is, $e+1\geq d+3>d+1$. Thus, $-s^{e+1}\neq -s^{d+1}$.

If $-s^{d+1}=-s^f$, then $d+1=f$. Thus, $R_3(G)$ and $R_3(H)$ can be written as follows:

$R_4(G)=-s^{d+1}-s^{e+1}+s^{d+3}+s^{e+3}+s^{d+10}+s^{d+10}$

$R_4(H)=-s^{e-1}-s^{d+1}+s^{e+1}+s^{d+3}+s^{d+8}+s^{d+12}$.

After simplifying $R_4(G)$ and $R_4(H)$, we have

$R_5(G)=-s^{e+1}+s^{e+3}+s^{d+10}+s^{d+10}$

$R_5(H)=-s^{e-1}+s^{e+1}+s^{d+8}+s^{d+12}$.

Thus, we have

$-s^{e+1}+s^{e+3}+s^{d+10}+s^{d+10}=-s^{e-1}+s^{e+1}+s^{d+8}+s^{d+12}$.

Therefore, we have $e=d+9$. At this point, we acquire the following equations: $e=d+9$, $f'=e-1=d+8$, $d'=f+1=d+2$, $e'=d$. Let $d=i$. Therefore, we obtain the solution, where G is isomorphic to $K_4(1,2,8,i,i+9,i+1)$ and H is isomorphic to $K_4(1,2,8,i+2,i,i+8)$.

Subcase 4.3. If min $\{d,e,f\}=d$ and min $\{d',e',f'\}=f'$, then $d=f'$. Note that max $\{e'+8,f'+9,d'+10\}=f'+9$. So, $f'+9\geq d'+10$. This contradicts min $\{d',e',f'\}=f'$.

Case 5. If max $\{e+8,f+9,d+10\}=f+9$ and max $\{e'+8,f'+9,d'+10\}=d'+10$, then $f+9=d'+10$, that is,

$$d'=f-1 \tag{3.3}$$

from $d+e+f=d'+e'+f'$, we have

$$e+d+1=e'+f'. \tag{3.4}$$

Consider the l.r.p. in $R_1(G)$ and the l.r.p. in $R_1(H)$, where min $\{d,e,f\}$ = min $\{d',e',f'\}$. Without loss of generality, let min $\{d,e,f\}=d$. The following subcases need to be considered.

Subcase 5.1. If min $\{d, e, f\} = d$ and min $\{d', e', f'\} = d'$, then we deal with this case the same way with case 1. So, we get $G \cong H$.
Subcase 5.2. If min $\{d, e, f\} = d$ and min $\{d', e', f'\} = e'$, then $d = e'$. From Eq. (3.4), we have $f' = e + 1$. Thus, we can write $R_1(G)$ and $R_1(H)$ as follows:

$R_6(G) = -s^d - s^e - s^f - s^{e+1} - s^{f+1} + s^{f+2} + s^{e+3} + s^{e+8} + s^{f+9} + s^{d+10}$,

$R_6(H) = -s^{f-1} - s^d - s^{e+1} - s^{d+1} - s^{e+2} + s^{e+3} + s^{d+3} + s^{d+8} + s^{e+10} + s^{f+9}$.

After simplifying $R_6(G)$ and $R_6(H)$, we have

$R_7(G) = -s^e - s^f - s^{f+1} + s^{f+2} + s^{e+8} + s^{d+10}$,

$R_7(H) = -s^{f-1} - s^{d+1} - s^{e+2} + s^{d+3} + s^{d+8} + s^{e+10}$.

Consider the term $-s^{d+1}$ in $R_7(H)$. Since max $\{e + 8, f + 9, d + 10\} = f + 9$, we have $f + 9 \geq d + 10$, that is, $f + 1 \geq d + 2 > d + 1$. So, $f + 1 \neq d + 1$. Thus, $-s^{d+1}$ in $R_7(H)$ must be equal to $-s^e$ or $-s^f$ in $R_7(G)$. If $-s^{d+1} = -s^f$, then $d + 1 = f$. From Eq. (3.3), we have $d = d'$ and

$R_8(G) = -s^e - s^{d+1} - s^{d+2} + s^{d+3} + s^{e+8} + s^{d+10}$,

$R_8(H) = -s^d - s^{d+1} - s^{e+2} + s^{d+3} + s^{d+8} + s^{e+10}$.

It is easy to see that $d = e$. Note that $d = e'$, so $e = e'$. From $d+e+f = d'+e'+f'$, we have $f = f'$. Thus, $G \cong H$.

If $-s^{d+1} = -s^e$, then $d + 1 = e$ and

$R_9(G) = -s^{d+1} - s^f - s^{f+1} + s^{f+2} + s^{d+9} + s^{d+10}$,

$R_9(H) = -s^{f-1} - s^{d+1} - s^{d+3} + s^{d+3} + s^{d+8} + s^{d+11}$.

After simplifying, we have

$-s^f - s^{f+1} + s^{f+2} + s^{d+9} + s^{d+10} = -s^{f-1} + s^{d+8} + s^{d+11}$

Thus, we have $f = d+9$. We also have the equations $e = d+1$, $e' = d$, $f' = e+1 = d + 2$ and $d' = f - 1 = d + 8$. Let $d = i$, then $f = i + 9$, $e = i + 1$, $e' = i$, $f' = i + 2$ and $d' = i + 8$. Thus, we obtain the solution, where $G \cong K_4(1, 2, 8, i, i + 1, i + 9)$ and $H \cong K_4(1, 2, 8, i + 8, i, i + 2)$.
Subcase 5.3. If min $\{d, e, f\} = d$ and min $\{d', e', f'\} = f'$, then $d = f'$. From Eq. (3.4), $e' = e + 1$. Note that Eq. (3.3) is $f = d' + 1$. We can write $R_1(G)$ and $R_1(H)$ as follows:

$R_{10}(G) = -s^d - s^e - s^f - s^{e+1} - s^{f+1} + s^{f+2} + s^{e+3} + s^{e+8} + s^{f+9} + s^{d+10}$,

$R_{10}(H) = -s^{f-1} - s^{e+1} - s^d - s^{e+2} - s^{d+1} + s^{d+2} + s^{e+4} + s^{e+9} + s^{d+9} + s^{f+9}$.

After simplifying $R_{10}(G)$ and $R_{10}(H)$, we have

$R_{11}(G) = -s^e - s^f - s^{f+1} + s^{f+2} + s^{e+3} + s^{e+8} + s^{d+10}$,

$R_{11}(H) = -s^{f-1} - s^{e+2} - s^{d+1} + s^{d+2} + s^{e+4} + s^{e+9} + s^{d+9}$.

For the same reasons stated in subcase 5.2, $-s^{d+1}$ must be equal to $-s^e$ or $-s^f$ in $R_{11}(G)$. If $-s^{d+1} = -s^e$, then $d+1 = e$. We can write $R_{11}(G)$ and $R_{11}(H)$ as follows:

$R_{12}(G) = -s^{d+1} - s^f - s^{f+1} + s^{f+2} + s^{d+4} + s^{d+9} + s^{d+10}$,

$R_{12}(H) = -s^{f-1} - s^{d+3} - s^{d+1} + s^{d+2} + s^{d+5} + s^{d+10} + s^{d+9}$.

After simplifying, we have

$-s^f - s^{f+1} + s^{f+2} + s^{d+4} = -s^{f-1} - s^{d+3} + s^{d+2} + s^{d+5}$.

So, we get $f = d + 3$. We also have $f' = d$, $e = d + 1$, $e' = e + 1 = d + 2$, $d' = f - 1 = d + 2$. Let $d = i$, then $e = i + 1$, $f = i + 3$, $d' = i + 2$, $e' = i + 2$, $f' = i$. Therefore, we obtain the solution, where $G \cong K_4(1, 2, 8, i, i + 1, i + 3)$ and $H \cong K_4(1, 2, 8, i + 2, i + 2, i)$.

Case 6. If max $\{e+8, f+9, d+10\} = e+8$ and max $\{e'+8, f'+9, d'+10\} = d'+10$, then $e+8 = d'+10$, that is,

$$d' = e - 2 \tag{3.5}$$

from $d + e + f = d' + e' + f'$, we have

$$d + f + 2 = e' + f'. \tag{3.6}$$

Consider the l.r.p. in $R_1(G)$ and the l.r.p. in $R_1(H)$. We have min $\{d, e, f\}$ = min $\{d', e', f'\}$. Without loss of generality, let min $\{d, e, f\} = d$. The following subcases need to be considered.

Subcase 6.1. If min $\{d, e, f\} = d$ and min $\{d', e', f'\} = d'$, then $d = d'$ and we can deal with this case the same way as Case 1. Thus, we get $G \cong H$.

Subcase 6.2. If min $\{d, e, f\} = d$ and min $\{d', e', f'\} = e'$, then $d = e'$. From Eq. (3.6), we have $f' = f + 2$. Thus, we can write $R_1(G)$ and $R_1(H)$ as follows:

$R_{13}(G) = -s^d - s^e - s^f - s^{e+1} - s^{f+1} + s^{f+2} + s^{e+3} + s^{e+8} + s^{f+9} + s^{d+10}$,

$R_{13}(H) = -s^{e-2} - s^d - s^{f+2} - s^{d+1} - s^{f+3} + s^{f+4} + s^{d+3} + s^{d+8} + s^{f+11} + s^{e+8}$.

After simplifying $R_{13}(G)$ and $R_{13}(H)$, we have

$R_{14}(G) = -s^e - s^f - s^{e+1} - s^{f+1} + s^{f+2} + s^{e+3} + s^{f+9} + s^{d+10}$,

$R_{14}(H) = -s^{e-2} - s^{f+2} - s^{d+1} - s^{f+3} + s^{f+4} + s^{d+3} + s^{d+8} + s^{f+11}$.

Consider the term $-s^{d+1}$ in $R_{14}(H)$. Since min $\{d, e, f\} = d$, $-s^{d+1}$ cannot cancel any negative term in $R_{14}(H)$. From max $\{e + 8, f + 9, d + 10\} = e + 8$, we have $e + 8 \geq d + 10$, that is $e + 1 \geq d + 3 > d + 1$. So, $-s^{d+1} \neq -s^{e+1}$. Moreover, $e \geq d + 2 > d + 1$, thus, $e \neq d + 1$, that is $-s^e \neq -s^{d+1}$. So, $-s^{d+1}$ must be equal to $-s^f$ or $-s^{f+1}$ in $R_{14}(G)$.

If $-s^{d+1} = -s^{f+1}$, then $d = f$. So, we have

$R_{15}(G) = -s^e - s^d - s^{e+1} - s^{d+1} + s^{d+2} + s^{e+3} + s^{d+9} + s^{d+10}$,

$R_{15}(H) = -s^{e-2} - s^{d+2} - s^{d+1} - s^{d+3} + s^{d+4} + s^{d+3} + s^{d+8} + s^{d+11}$.

After simplifying, consider the h.r.p. in $R_{15}(G)$ and the h.r.p. in $R_{15}(H)$. We have $s^{e+3} = s^{d+11}$, that is $e + 3 = d + 11$. This contradicts $R_{15}(G) = R_{15}(H)$ since $-s^e$ cannot be cancelled by $+s^{d+8}$ in $R_{15}(H)$.

If $-s^{d+1} = -s^f$, then $d + 1 = f$. Thus, we have

$R_{16}(G) = -s^e - s^{d+1} - s^{e+1} - s^{d+2} + s^{d+3} + s^{e+3} + s^{d+10} + s^{d+10}$,

$R_{16}(H) = -s^{e-2} - s^{d+3} - s^{d+1} - s^{d+4} + s^{d+5} + s^{d+3} + s^{d+8} + s^{d+12}$.

After simplifying, consider the h.r.p. in $R_{16}(G)$ and h.r.p. in $R_{16}(H)$. We have $s^{e+3} = s^{d+12}$. The term s^{d+8} in $R_{16}(H)$ cannot be cancelled since there is no term equal to it. This contradicts $R_{16}(G) = R_{16}(H)$.

Subcase 6.3. If min $\{d, e, f\} = d$ and min $\{d', e', f'\} = f'$, then $d = f'$. From Eq. (3.6), $e' = f + 2$ and note that from Eq. (3.5), $d' = e - 2$. Thus, we have

$R_{17}(G) = -s^d - s^e - s^f - s^{e+1} - s^{f+1} + s^{f+2} + s^{e+3} + s^{e+8} + s^{f+9} + s^{d+10}$,

$R_{17}(H) = -s^{e-2} - s^{f+2} - s^d - s^{f+3} - s^{d+1} + s^{d+2} + s^{f+5} + s^{f+10} + s^{d+9} + s^{e+8}$.

After simplifying, consider the term $-s^{d+1}$ in $R_{17}(H)$. For the same reasons stated in subcase 4.2, $-s^{d+1}$ can only be equal to $-s^f$ or $-s^{f+1}$ in $R_{17}(G)$.

If $-s^{d+1} = -s^f$, then $d + 1 = f$. So, we have

$R_{18}(G) = -s^e - s^{d+1} - s^{e+1} - s^{d+2} + s^{d+3} + s^{e+3} + s^{d+10} + s^{d+10}$,

$R_{18}(H) = -s^{e-2} - s^{d+3} - s^{d+4} - s^{d+1} + s^{d+2} + s^{d+6} + s^{d+11} + s^{d+9}$.

After simplifying, consider the h.r.p. in $R_{18}(G)$ and the h.r.p. in $R_{18}(H)$. We have $s^{e+3} = s^{d+11}$. So, $e + 3 = d + 11$, thus $e = d + 8$. There is no term s^{d+8} which is equal to the term s^e in $R_{18}(G)$. This contradicts $R_{18}(G) = R_{18}(H)$.
If $-s^{d+1} = -s^{f+1}$, then $d + 1 = f + 1$, that is $d = f = f'$. This case is the same as case 1. So, we get the same result $G \cong H$. At this point, we have solved the equation $R(G) = R(H)$ and the solution is as follows:

$$K_4(1, 2, 8, i + 9, i, i + 1) \sim K_4(1, 2, 8, i + 2, i, i + 8),$$
$$K_4(1, 2, 8, i, i + 1, i + 9) \sim K_4(1, 2, 8, i + 8, i, i + 2),$$
$$K_4(1, 2, 8, i, i + 1, i + 3) \sim K_4(1, 2, 8, i + 2, i + 2, i),$$

where $i \geq 1$. The proof is now complete. □

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REFERENCES

[1] S. Catada-Ghimire, H. Roslan, Y.H. Peng, *On chromatic uniqueness of a family of K_4-homeomorphs*, Bull. Malays. Math. Sci. Soc. (2) **32** (2009) 3, 327–333.

[2] S. Catada-Ghimire, H. Roslan, Y.H. Peng, *On chromatic equivalence of a family of K_4-homeomorphs*, Far-East Journal of Mathematical Sciences **30** (2008) 2, 389–400.

[3] S. Catada-Ghimire, H. Roslan, Y.H. Peng, *On chromatic equivalence of K_4-homeomorphs II*, Advances and Application in Discrete Mathematics **3** (2009) 1, 53–66.

[4] S. Catada-Ghimire, H. Roslan, Y.H. Peng, *On chromatic equivalence of K_4-homeomorphs*, Southeast Asian Bulletin of Mathematics **33** (2009), 749–757.

[5] X.E. Chen, K.Z. Ouyang, *Chromatic classes of certain 2-connected $(n, n + 2)$-graphs homeomorphs to K_4*, Discrete Mathematics **172** (1997), 17–29.

[6] F.M. Dong, K.M. Koh, K.L. Teo, *Chromatic polynomials and chromaticity of graphs*, World Scientific Publishing Co. Ptd. Ltd., Singapore, 2005, 118–123.

[7] K.M. Koh, K.L. Teo, *The search for chromatically unique graphs*, Graph. Combin. **6** (1990), 259–285.

[8] W.M. Li, *Almost every K_4-homeomorphs is chromatically unique*, Ars Combin. **23** (1987), 13–36.

[9] Y.L. Peng, *Some new results on chromatic uniqueness of K_4-homeomorphs*, Discrete Mathematics **288** (2004), 177–183.

[10] Y.L. Peng, *Chromaticity of family of K_4-homeomorphs*, personal communication (2006).

[11] Y.L. Peng, *Chromatic uniqueness of a family of K_4-homeomorphs*, Discrete Mathematics **308** (2008), 6132–6140.

[12] Y.L. Peng, *On the chromatic equivalence classes of K_4-(1,3,3,δ,ϵ,η)*, J. Qinghai Normal Univ. **4** (2003), 1–3 [in Chinese].

[13] Y.L. Peng, Zengyi, *On the chromatic equivalence of two families K_4-homeomorphs*, J. Suzhou Sci. Technol. Univ. **4** (2004), 31–34 [in Chinese].

[14] Y.L. Peng, *Chromatic equivalence of a family of K_4-homeomorphs*, J. Shanghai Normal Univ. (Natural Sciences) **33** (2004) 4, 9–11 [in Chinese].

[15] S. Xu, *A lemma in studying chromaticity*, Ars. Combin. **32** (1991), 315–318.

[16] C.Y. Chao, L.C. Zhao, *Chromatic polynomials of a family of graphs*, Ars. Combin. **15** (1983), 111–129.

[17] E.G. Whitehead Jr., L.C. Zhao, *Chromatic uniqueness and equivalence of K_4-homeomorphs*, Journal of Graph Theory **8** (1984), 355–364.

[18] W.M. Li, *Almost every K_4-homeomorphs is chromatically unique*, Ars Combin. **23** (1987), 13–36.

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