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FRACTIONAL NONLOCAL INTEGRODIFFERENTIAL EQUATIONS OF MIXED TYPE WITH TIME-VARYING GENERATING OPERATORS AND OPTIMAL CONTROL

Abstract. In this paper, a class of fractional integrodifferential equations of mixed type with time-varying generating operators and nonlocal conditions is considered. Using a contraction mapping principle and Krasnoselskii's fixed point theorem via Gronwall's inequailty, the existence and uniqueness of mild solution are given. The existence of optimal pairs of systems governed by fractional integrodifferential equations of mixed type with time-varying generating operators and nonlocal conditions is also presented.

Keywords: fractional integrodifferential equations of mixed type, time-varying generating operators, nonlocal conditions, fixed point theorem, existence, optimal control.

Mathematics Subject Classification: 45N05, 93C25.

1. INTRODUCTION

Fractional differential equations has recently been proved to be valuable tools in the modeling of many phenomena in various fields of engineering, physics, economy and science. We can find numerous applications in viscoelasticity, electrochemistry, control, porous media, electromagnetic, etc. [11, 13–15, 24, 25]. In recent years, there has been a significant development in fractional differential equations. One can see the monographs of Kilbas et al. [18], Miller and Ross [23], Podlubny [29], Lakshmikantham et al. [20], and the papers on abstract fractional differential equations [7–10, 12, 17, 21, 22, 26, 27] and the references therein. On the other hand, the study of initial value problems with nonlocal conditions arises to deal especially with some situations in physics. For comments and motivations of nonlocal Cauchy problems in different fields, we refer the reader to [1]–[6] and the references contained therein.

Very recently, the fractional differential equations with nonlocal conditions on infinite dimensional spaces were covered by authors such as Benchohra, Mophou, N'Guérékata, Sakthivel and etc. However, to our knowledge, the fractional integrodifferential equations of mixed type with time-varying generating operators and nonlocal initial conditions has not been discussed extensively. In particular, few authors discuss the optimal control problem of systems governed by the fractional integrodifferential equations on infinite dimensional spaces.

Our aim in this paper is to consider the following more general fractional integrodifferential equations of mixed type with time-varying generating operators and nonlocal initial conditions:

$$D^q x(t) = A(t)x(t) + t^n f\left(t, x(t), (Kx)(t), (Hx)(t)\right), \tag{1.1}$$
$$t \in J = [0,T],\ n \in Z^+,\ q \in (0,1),$$
$$x(0) = g(x) + x_0 \tag{1.2}$$

in a general Banach space $(X, \|\cdot\|)$, where $\{A(t),\, t \in J\}$ is a family of closed densely defined linear unbounded operators on X, $x_0 \in X$, f: $J \times X \times X \times X \to X$ is a nonlinear function, and $g : C(J,X) \to X$ constitutes a nonlocal Cauchy problem. The derivative D^q is understood here in the Riemann-Liouville sense. Operators K and H are nonlinear integral operators given by

$$\begin{aligned}(Kx)(t) &= \int_0^t k(t,s,x(s))ds,\\ (Hx)(t) &= \int_0^T h(t,s,x(s))ds.\end{aligned} \tag{1.3}$$

The rest of this paper is organized as follows. In Section 2, we give some preliminaries and introduce the mild solution of system (1.1)–(1.2). In Section 3, we study the existence and uniqueness of mild solutions for system (1.1)–(1.2) using the Banach contraction principle and Krasnoselskii's fixed point theorem visa Gronwall's inequality. At last, an existence result of optimal controls for a Lagrange problem (P) is proved.

2. PRELIMINARIES

Let $L_b(X)$ be the Banach space of all linear and bounded operators on X. $C(J,X)$ be the Banach space of all X-valued continuous functions from J into X endowed with the norm $\|x\|_C = \sup_{t\in J} \|x(t)\|$.

For the family $\{A(t),\, t \in J\}$ of linear operators, we need the following assumption.

[HA]: For $t \in J$, one has

(A_1) The domain $D(A(t)) = D$ is independent of t and is dense on X.

(A_2) For $t \geq 0$, the resolvent $R(\lambda, A(t)) = (\lambda I - A(t))^{-1}$ exists for all λ with $Re\lambda \leq 0$, and there is a constant $\widetilde{M} > 0$ independent of λ and t such that

$$\|R(\lambda, A(t))\| \leq \widetilde{M}(1 + |\lambda|)^{-1} \quad \text{for} \quad Re\lambda \leq 0.$$

(A_3) There exist constants $L > 0$ and $0 < \alpha \leq 1$ such that

$$\left\| \big(A(t) - A(\theta)\big) A^{-1}(\tau) \right\| \leq L|t - \theta|^{\alpha} \quad \text{for} \quad t, \theta, \tau \in [0, T].$$

(A_4) The resolvent $R(\lambda, A(t))(t \geq 0)$ is compact.

Lemma 2.1 ([19], Lemma 2.2 of [28])**.** *Under the assumption* [HA]*, the Cauchy problem*

$$\dot{x}(t) + A(t)x(t) = 0,\ t \in (0, T] \quad \textit{with} \quad x(0) = x_0 \tag{2.1}$$

has a unique evolution system $\{U(t, \theta) \mid 0 \leq \theta \leq t \leq T\}$ *on* X *satisfying the following properties:*

(1) $U(t, \theta) \in L_b(X)$ *for* $0 \leq \theta \leq t \leq T$*. Denotes* $M = \sup_{t \in J} \|U(t, s)\|_{L_b(X)}$*, which is a finite number.*
(2) $U(t, r)U(r, \theta) = U(t, \theta)$ *for* $0 \leq \theta \leq r \leq t \leq T$*.*
(3) $U(\cdot, \cdot)x \in C(\Lambda, X)$ *for* $x \in X$*,* $\Lambda = \{(t, \theta) \in J \times J \mid 0 \leq \theta \leq t \leq T\}$*.*
(4) *For* $0 \leq \theta < t \leq T$*,* $U(t, \theta)\colon X \longrightarrow D$ *and* $t \longrightarrow U(t, \theta)$ *is strongly differentiable on* X*. The derivative* $\frac{\partial}{\partial t} U(t, \theta) \in L_b(X)$ *and it is strongly continuous on* $0 \leq \theta < t \leq T$*. Moreover,*

$$\frac{\partial}{\partial t} U(t, \theta) = -A(t)U(t, \theta) \quad \textit{for} \quad 0 \leq \theta < t \leq T,$$
$$\left\| \frac{\partial}{\partial t} U(t, \theta) \right\|_{L_b(X)} = \|A(t)U(t, \theta)\|_{L_b(X)} \leq \frac{C}{t - \theta},$$
$$\left\| A(t)U(t, \theta)A(\theta)^{-1} \right\|_{L_b(X)} \leq C \quad \textit{for} \quad 0 \leq \theta \leq t \leq T.$$

(5) *For every* $v \in D$ *and* $t \in (0, T], U(t, \theta)v$ *is differentiable with respect to* θ *on* $0 \leq \theta \leq t \leq T$

$$\frac{\partial}{\partial \theta} U(t, \theta)v = U(t, \theta)A(\theta)v.$$

(6) $U(t, \theta)$ *is compact operator for* $0 \leq \theta < t \leq b$*.*
And, for each $x_0 \in X$*, the Cauchy problem* (2.1) *has a unique classical solution* $x \in C^1(J, X)$ *given by*

$$x(t) = U(t, 0)x_0, \quad t \in J.$$

Let us recall the following known definitions. For more details see [29].

Definition 2.2. The *Riemann-Liouville fractional integral operator of order* $\gamma \geq 0$ *of a function* $f \in C_\alpha$*,* $\alpha \geq -1$ is defined as

$$I^{\gamma} f(t) = \frac{1}{\Gamma(\gamma)} \int_0^t (t - s)^{\gamma - 1} f(s) ds,$$

where $\Gamma(\cdot)$ is the Euler gamma function.

Definition 2.3. If the function $f \in C^m_{-1}$, $m \in N$, *the fractional derivative of order* $\gamma > 0$ *of a function* $f(t)$ *in the Caputo sense* is given by

$$\frac{d^\gamma f(t)}{dt^\gamma} = \frac{1}{\Gamma(m-\gamma)} \int_0^t (t-s)^{m-\gamma-1} f^{(m)}(s) ds, \; m-1 < \gamma \leq m.$$

Based on Definition 3.1 of [27], we can introduce the mild solution of system (1.1)–(1.2).

Definition 2.4. A *mild solution of system* (1.1)–(1.2) is a function in $C(J, X)$ such that

$$x(t) = U(t,0)(x_0 + g(x)) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} s^n U(t,s) f\left(s, x(s), (Kx)(s), (Hx)(s)\right) ds.$$

3. MAIN RESULTS

In this section, we give the existence and uniqueness of the mild solutions for system (1.1)–(1.2).

We need the following assumptions.

[Hf]: $f : J \times X \times X \times X \to X$ is continuous and there exists a function $m_1, m_2, m_3 \in L^1_{Loc}(J, R^+)$ such that

$$\|f(t, x_1, x_2, x_3) - f(t, y_1, y_2, y_3)\| \leq m_1(t)\|x_1 - y_1\| + m_2(t)\|x_2 - y_2\| + m_3(t)\|x_3 - y_3\|$$

for all $x_i, y_i \in X$, $i = 1, 2, 3$ and $t \in J$.

[Hk]: Let $D_k = \{(t,s) \in R^2; 0 \leq s \leq t \leq T\}$. The function $k : D_k \times X \to X$ is continuous and there exists a $m_k(t,s) \in C(D_k, R^+)$ and

$$K^* = \sup_{t \in J} \int_0^t m_k(t,s) ds < \infty$$

such that

$$\|k(t,s,x) - k(t,s,y)\| \leq m_k(t,s)\|x - y\|$$

for each $(t,s) \in D_k$ and $x, y \in X$.

[Hh]: Let $D_h = \{(t,s) \in R^2; 0 \leq s, t \leq T\}$. The function $h : D_h \times X \to X$ is continuous and there exists a $m_h(t,s) \in C(D_h, R^+)$ and

$$H^* = \sup_{t \in J} \int_0^T m_h(t,s) ds < \infty$$

such that

$$\|h(t,s,x) - h(t,s,y)\| \leq m_h(t,s)\|x - y\|$$

for each $(t,s) \in D_h$ and $x, y \in X$.

[Hg]: $g : C(J,X) \to X$ is a continuous and there exists a constant $l_g > 0$ such that

$$\|g(x) - g(y)\| \le l_g \|x - y\|_C,$$

for arbitrary $x, y \in C(J,X)$, where $\|\cdot\|_C$ denotes $\|\cdot\|_{C(J,X)}$.

[HΩ_n]: The function $\Omega_n : J \to R^+$, $n \in Z^+$, defined by

$$\Omega_n = M\left[l_g + \frac{t^n T^q}{(n+1)\Gamma(q)}\left(\|m_1\|_{L^1_{Loc}(J,R^+)} + K^*\|m_2\|_{L^1_{Loc}(J,R^+)} + H^*\|m_3\|_{L^1_{Loc}(J,R^+)}\right)\right],$$

satisfies $0 < \Omega_n \le \omega < 1$, for all $t \in J$.

Now we are ready to give our first existence and uniqueness result which is based on the Banach contraction mapping principle.

Theorem 3.1. *Assume that the conditions* (A_1)–(A_3), [Hf], [Hk], [Hh], [Hg] *and* [HΩ_n] *are satisfied. Then system* (1.1)–(1.2) *has a unique mild solution.*

Proof. We consider the operator $\Gamma : C(J,X) \to C(J,X)$ defined by

$$(\Gamma x)(t) = U(t,0)[x_0 + g(x)] + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} s^n U(t,s) f\left(s, x(s), (Kx)(s), (Hx)(s)\right) ds, \tag{3.1}$$

for all $t \in J$. Note that Γ is well defined on $C(J,X)$.

Now, take $t \in J$ and $x, y \in C(J,X)$. Then we have

$$\|\Gamma x(t) - \Gamma y(t)\| \le \|U(t,0)(g(x) - g(y))\| + \frac{1}{\Gamma(q)} \times$$
$$\times \left\| \int_0^t (t-s)^{q-1} s^n U(t,s)[f(s, x(s), (Kx)(t), (Hx)(t)) - f(s, y(s), (Ky)(t), (Hy)(t))] ds \right\|.$$

Hence, we get

$$\|\Gamma x(t) - \Gamma y(t)\| \le M\|g(x) - g(y)\| +$$
$$+ M\frac{T^{q-1}}{\Gamma(q)} \int_0^t s^n \|f(s, x(s), (Kx)(t), (Hx)(t)) - f(s, y(s), (Ky)(t), (Hy)(t))\| ds.$$

According to [Hf] and [Hg], we obtain

$$
\begin{aligned}
\|\Gamma x(t)-\Gamma y(t)\| \le{}& Ml_g\|x-y\|_C + M\frac{T^{q-1}}{\Gamma(q)}\int_0^t s^n m_1(s)\|x(s))-y(s)\|ds+\\
&+M\frac{T^{q-1}}{\Gamma(q)}\int_0^t s^n m_2(s)\|(Kx)(s))-(Ky)(s)\|ds+\\
&+M\frac{T^{q-1}}{\Gamma(q)}\int_0^t s^n m_3(s)\|(Hx)(s))-(Hy)(s)\|ds \le\\
\le{}& Ml_g\|x-y\|_C+\\
&+M\frac{T^{q-1}}{\Gamma(q)}\|x-y\|_C\int_0^t s^n[m_1(s)+K^*m_2(s)+H^*m_3(s)]ds.
\end{aligned}
$$

Therefore, we can deduce that

$$
\begin{aligned}
\|\Gamma x(t)-\Gamma y(t)\| \le{}& M\Bigg[l_g+\frac{t^{n+1}T^{q-1}}{(n+1)\Gamma(q)}\Big(\|m_1\|_{L^1_{Loc}(J,R^+)}+K^*\|m_2\|_{L^1_{Loc}(J,R^+)}+\\
&+H^*\|m_3\|_{L^1_{Loc}(J,R^+)}\Big)\Bigg]\|x-y\|_C \le\\
\le{}& \Omega_n(t)\|x-y\|_C.
\end{aligned}
$$

Thus, we obtain

$$\|\Gamma x-\Gamma y\|_C \le \Omega_n(t)\|x-y\|_C.$$

Hence, assumption $[\mathrm{H}\Omega_n]$ allows us to conclude in view of the contraction mapping principle, that Γ has a unique fixed point $x \in C(J,X)$, and

$$x(t)=U(t,0)[x_0+g(x)]+\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}s^nU(t,s)f(s,x(s),(Kx)(s),(Hx)(s))ds$$

which is the unique mild solution of system (1.1)–(1.2). □

Our second result uses the following well known fixed point theorem.

Theorem 3.2 (Krasnoselskii)**.** *Let $\mathfrak{B}$ be a closed convex and nonempty subsets of a Banach space X. Suppose that $\mathcal{L}$ and $\mathcal{N}$ are in general nonlinear operators which map $\mathfrak{B}$ into X such that:*

(1) *$\mathcal{L}x+\mathcal{N}y \in \mathfrak{B}$ whenever $x,y \in \mathfrak{B}$;*
(2) *$\mathcal{L}$ is a contraction mapping;*
(3) *$\mathcal{N}$ is compact and continuous.*

Then there exists $z \in \mathfrak{B}$ such that $z=\mathcal{L}z+\mathcal{N}z$.

Suppose that

[Hf′]: $f : J\times X\times X\times X \to X$, for a.e. $t \in J$, the function $f(t, \cdot, \cdot, \cdot) : X\times X\times X \to X$ is continuous and for all $x, y, z \in X$, the function $f(\cdot, x, y, z) : J \to X$ is measurable. There exists a function $\rho \in L^1_{Loc}(J, R^+)$ such that

$$\|f(t, x, y, z)\| \le \rho(t), \text{ for all } t \in J, x, y, z \in X.$$

[Hk′]: The function $k : D_k \times X \to X$ is continuous and there exists $m_k \in L^1_{Loc}(J, R^+)$ such that

$$\|k(t, s, x)\| \le m_k(t)$$

for each $(t, s) \in D_k$ and $x \in X$.

[Hh′]: The function $h : D_h \times X \to X$ is continuous and there exists $m_h \in L^1_{Loc}(J, R^+)$ such that

$$\|h(t, s, x)\| \le m_h(t)$$

for each $(t, s) \in D_h$ and $x \in X$.

Now we are ready to state and prove the following existence result.

Theorem 3.3. *Assume that the conditions* (A_1)-(A_3), [Hf′], [Hk′], [Hh′], [Hg] *are satisfied. Then system* (1.1)–(1.2) *has at least one mild solution on J provided that* $Ml_g < 1$.

Proof. Let us choose

$$r = M(\|x_0\| + G) + M\frac{T^{n+q}}{(n + 1)\Gamma(q)}\|\rho\|_{L^1_{Loc}(J,R^+)} + Mc_0 + c_2\frac{T^{n+q}}{n + 1}$$

with

$$G = \sup_{x\in C(J,X)} \{\|g(x)\|\}, \tag{3.2}$$

c_0 and c_2 defined respectively by (3.3) and (3.4) below.

Consider the ball

$$B_r = \{x \in C(J, X) \mid \|x\| \le r\}.$$

Define on B_r the operators Γ_1 and Γ_2 by

$$(\Gamma_1 x)(t) = U(t, 0)[x_0 + g(x)],$$

and

$$(\Gamma_2 x)(t) = \frac{1}{\Gamma(q)}\int_0^t (t - s)^{q-1}s^n U(t, s)f(s, x(s), (Kx)(s), (Hx)(s))ds.$$

Step 1. Let us observe that if $x, y \in B_r$ then $\Gamma_1 x + \Gamma_2 y \in B_r$.

In fact,

$$\|(\Gamma_1 x)(t) + (\Gamma_2 y)(t)\| \le M\|x_0 + g(x)\| + \\ + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} s^n \|f(s,y(s),(Ky)(s),(Hy)(s))\| ds \le \\ \le M(\|x_0\| + \|g(x)\|) + \\ + M\frac{T^{q-1}}{\Gamma(q)}\int_0^t s^n \|f(s,y(s),(Ky)(s),(Hy)(s))\| ds,$$

which according to (3.2), gives

$$\|(\Gamma_1 x)(t) + (\Gamma_2 y)(t)\| \le M(\|x_0\| + G) + M\frac{T^{n+q}}{(n+1)\Gamma(q)}\|\rho\|_{L^1_{Loc}(J,R^+)} \le r.$$

Hence, we can deduce that

$$\|\Gamma_1 x + \Gamma_2 x\|_C \le r.$$

Step 2. We show that Γ_1 is a contraction mapping.

For any $t \in J$, $x, y \in C(J,X)$ we have

$$\|(\Gamma_1 x)(t) - (\Gamma_1 y)(t)\| \le M\|g(x) - g(y)\|$$

which in view of [Hg], gives

$$\|(\Gamma_1 x)(t) - (\Gamma_1 y)(t)\| \le Ml_g\|x - y\|_C,$$

which implies that

$$\|\Gamma_1 x - \Gamma_1 y\|_C \le Ml_g\|x - y\|_C.$$

Since $Ml_g < 1$, then Γ_1 is a contraction mapping.

Step 3. Let us prove that Γ_2 is continuous and compact.

For this purpose, we assume that $x_n \to x$ in $C(J,X)$. It comes from the continuity of k and h that

$$k(t,s,x_n(s)) \to k(t,s,x(s)) \quad \text{and} \quad \|k(t,s,x_n(s))\| \le m_k(t).$$

$$h(t,s,x_n(s)) \to h(t,s,x(s)) \quad \text{and} \quad \|h(t,s,x_n(s))\| \le m_h(t).$$

By the dominated convergence theorem,

$$\int_0^t h(t,s,x_n(s))ds \to \int_0^t h(t,s,x(s))ds, \quad \int_0^T k(t,s,x_n(s))ds \to \int_0^T k(t,s,x(s))ds, \text{ as } n \to \infty.$$

Then by [Hf′], we have

$$f(s,x_n(s),(Kx_n)(s),(Hx_n)(s)) \to f(s,x(s),(Kx)(s),(Hx)(s)) \text{ as } n \to \infty,\ s \in J.$$

$$\|f(s, x_n(s), (Kx_n)(s), (Hx_n)(s))\| \leq \rho(s),\ \ s \in J.$$

By the dominated convergence theorem again, we have

$$\|(\Gamma_2 x_n)(t) - (\Gamma_2 x)(t)\| \leq$$
$$\leq M\frac{T^{n+q}}{(n+1)\Gamma(q)}\int_0^t \|f(s, x_n(s), (Kx_n)(s), (Hx_n)(s)) - f(s, x(s), (Kx)(s), (Hx)(s))\| ds$$
$$\to 0,\ \text{as } n \to \infty,$$

which implies that Γ_2 is continuous.

To prove that Γ_2 is a compact operator, we observe that Γ_2 is a composition of two operators, that is, $\Gamma_2 = \mathcal{U} \circ \mathcal{V}$, where

$$(\mathcal{V}x)(s) = U(t,s)f(s, x(s), (Kx)(s), (Hx)(s)),\ t \in J,\ 0 < s < t,$$

and

$$(\mathcal{U}y)(t) = \int_0^t (t-s)^{q-1} s^n y(s) ds,\ t \in J.$$

Since for the same reason as Γ_2, the operator $\mathcal{V}$ is also continuous, it suffices to prove that $\mathcal{V}$ is uniformly bounded and $\mathcal{U}$ is compact to prove that Γ_2 is compact.

Let $x \in B_r$. Then $(Kx)(t) \in B_r^{'} = \{v \in C(J, X) \mid \|v\|_C \leq \|m_k\|_{L^1_{Loc}(J,R^+)} \equiv \hat{K}^* r\}$ and $(Hx)(t) \in B_r^{''} = \{v \in C(J, X) \mid \|v\|_C \leq \|m_h\|_{L^1_{Loc}(J,R^+)} \equiv \hat{H}^* r\}$. In view of [Hf′], f is bounded on the compact set $J \times B_r \times B_r^{'} \times B_r^{''}$. Therefore, we set

$$c_0 = \sup_{(t,x,y,z) \in J \times B_r^{'} \times B_r^{''}} \|f(t, x(t), y(t), z(t))\| < \infty. \tag{3.3}$$

Then, using (3.3), we get

$$\|(\mathcal{V}x)(s)\| \leq \|U(t,s)\| \|f(s, x(s), (Kx)(s), (Hx)(s))\| \leq Mc_0 \leq r,$$

from which we deduce that $\|\mathcal{V}x\|_C \leq r$. This means that $\mathcal{V}$ is uniformly bounded on B_r.

Since $y \in C(J, X)$, we set

$$c_2 = \sup_{t \in J} \|y(t)\| < \infty. \tag{3.4}$$

Then, on the other hand, we have

$$\|(\mathcal{U}y)(t)\| = \left\| \int_0^t (t-s)^{q-1} s^n y(s) ds \right\| \leq c_2 \int_0^t (t-s)^{q-1} s^n ds \leq c_2 \frac{T^{n+q}}{n+1} \leq r$$

and on the other hand, for $0 < s < t_2 < t_1 < T$,

$$\begin{aligned}
\|(\mathcal{U}y)(t_1) - (\mathcal{U}y)(t_2)\| &= \left\| \int_0^{t_1} (t_1 - s)^{q-1} s^n y(s) ds - \int_0^{t_2} (t_2 - s)^{q-1} s^n y(s) ds \right\| \leq \\
&\leq \left\| \int_0^{t_2} [(t_1 - s)^{q-1} - (t_2 - s)^{q-1}] s^n y(s) ds \right\| + \\
&\quad + \left\| \int_{t_1}^{t_2} (t_1 - s)^{q-1} s^n y(s) ds \right\| \leq \\
&\leq \int_0^{t_2} |(t_1 - s)^{q-1} - (t_2 - s)^{q-1}| s^n \|y(s)\| ds + \\
&\quad + \int_{t_1}^{t_2} (t_1 - s)^{q-1} s^n \|y(s)\| ds \leq \\
&\leq \frac{c_2 T^n}{q} |2(t_1 - t_2)^q + t_2^q - t_1^q| \leq \\
&\leq 2 \frac{c_2 T^n}{q} |t_1 - t_2|^q,
\end{aligned}$$

which does not depend on y. So $\mathcal{U}B_r$ is relatively compact. By the Arzela-Ascoli Theorem, $\mathcal{U}$ is compact. In short, we have proven that Γ_2 is continuous and compact, Γ_1 is a contraction mapping and $\Gamma_1 x + \Gamma_2 y \in B_r$ if $x, y \in B_r$. Hence, the Krasnoselskii theorem allows us to conclude that system (1.1)–(1.2) has at least one mild solution on J. □

Corollary 3.4. *In addition to the assumptions of Theorem 3.3, assumptions* [Hf], [Hk], [Hh] *also hold. Then system* (1.1)–(1.2) *has a unique mild solution on* J.

Proof. To prove the uniqueness of $x(\cdot)$, let $y(\cdot)$ be another mild solution of system (1.1)–(1.2) with nonlocal condition $y_0 + g(y)$. It comes from

$$\begin{aligned}
&\|x(t) - y(t)\| \leq \|U(t,0)x_0 - U(t,0)y_0\| + \|U(t,0)g(x) - U(t,0)g(y)\| + \\
&+ \frac{1}{\Gamma(q)} \int_0^t \|(t-s)^{q-1} s^n\| \|U(t,s)[f(s, x(s), (Kx)(s), (Hx)(s)) - \\
&\qquad - f(s, y(s), (Ky)(s), (Hy)(s))]\| ds \leq \\
&\leq M\|x_0 - y_0\| + M l_g \|g(x) - g(y)\| + \\
&+ M \frac{T^{n+q}}{(n+1)\Gamma(q)} \int_0^t (m_1(t)T + m_2(t)K^*T + m_3(t)H^*T) \|x(s) - y(s)\| ds
\end{aligned}$$

that

$$\|x(t)-y(t)\| \leq \frac{M}{1-Ml_g}\|x_0-y_0\|+$$
$$+MT\frac{T^{n+q}}{(n+1)\Gamma(q)(1-Ml_g)}\int_0^t (m_1(t)+m_2(t)K^*+m_3(t)H^*)\|x(s)-y(s)\|ds \leq$$
$$\leq \frac{M}{1-Ml_g}\|x_0-y_0\|+\widehat{M}MT\frac{T^{n+q}}{(n+1)\Gamma(q)(1-Ml_g)}\int_0^t \|x(s)-y(s)\|ds,$$

where $\widehat{M}=\|m_1\|_{L^1_{Loc}(J,R^+)}+K^*\|m_2\|_{L^1_{Loc}(J,R^+)}+H^*\|m_3\|_{L^1_{Loc}(J,R^+)}$, which implies by Gronwall's inequality that

$$\|x(t)-y(t)\| \quad \leq \quad \widetilde{M}\frac{M}{1-Ml_g}\|x_0-y_0\|, \widetilde{M}>0,$$

which yield the uniqueness of $x(\cdot)$. □

4. EXISTENCE OF OPTIMAL CONTROLS

We suppose that Y is another separable reflexive Banach space from which the controls u take the value. We denote a class of nonempty closed and convex subsets of Y by $W_f(Y)$. The multifunction $\omega : J \longrightarrow W_f(Y)$ is measurable and $\omega(\cdot) \subset E$, where E is a bounded set of Y, the admissible control set $U_{ad} = S^p_\omega = \{u \in L^p(E) \mid u(t) \in \omega(t)\ a.e.\}$, $1 < p < \infty$. Then $U_{ad} \neq \emptyset$ (see P142 Proposition 1.7 and P174 Lemma 3.2 of [16]).

Consider the following controlled system

$$\begin{cases} D^q x(t) = A(t)x(t) + t^n f(t, x(t), (Kx)(t), (Hx)(t)) + C(t)u(t), \\ \qquad\qquad t \in J,\ u \in U_{ad},\ n \in Z^+,\ q \in (0,1), \\ x(0) = g(x) + x_0. \end{cases} \tag{4.1}$$

Assumption [HC]: $C \in L_\infty(J,L(Y,X))$.

It is easy to see that $Cu \in L^p(J,X)$ for all $u \in U_{ad}$. Define $\widetilde{f}(t,x) = f(t,x(t),(Kx)(t),(Hx)(t)) + C(t)u(t)$. It is obvious that $\widetilde{f}$ satisfies the assumption [Hf′].

Theorem 4.1. *In addition to assumptions of Theorem 3.3, suppose assumption* [HC] *holds. For every $u \in U_{ad}$, system (4.1) has a mild solution corresponding to u given by the solution of the following integral equation*

$$x^u(t) = U(t,0)[x_0 + g(x)] + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} s^n U(t,s) f(s, x(s), (Kx)(s), (Hx)(s)) ds +$$

$$+ \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} s^n U(t,s) C(s) u(s) ds.$$

Proof. Compared with Theorem 3.3, the key step is to check the term containing control policy. Consider

$$\xi(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} s^n U(t,s) C(s) u(s) ds,$$

using the Hölder inequality, we have

$$\begin{aligned}
\|\xi(t)\| &\le M \frac{T^{q-1}}{\Gamma(q)} \int_0^t s^n \|C(s)u(s)\| ds \le \\
&\le M \|C\|_\infty \frac{T^{n+q}}{(n+1)\Gamma(q)} \int_0^t \|u(s)\|_Y ds \le \\
&\le M \|C\|_\infty \frac{T^{n+q}}{(n+1)\Gamma(q)} \left(\int_0^t 1^{\frac{p-1}{p}} ds \right)^{\frac{p-1}{p}} \left(\int_0^t \|u(s)\|_Y^p ds \right)^{\frac{1}{p}} \le \\
&\le M \|C\|_\infty \frac{T^{n+q}}{(n+1)\Gamma(q)} \|u\|_{L^p(J,Y)},
\end{aligned}$$

where $\|C\|_\infty$ is the norm of operator C in the Banach space $L_\infty(J, L(Y,X))$. It is easy to see that $\|U(t,\cdot)C(\cdot)u(\cdot)\|$ is integrable. Hence $\xi(\cdot) \in C(J,X)$. Using Theorem 3.3, one can verify it immediately. □

Assumption [HL]:

[HL1] The functional $l : J \times X \times Y \longrightarrow R \cup \{\infty\}$ is Borel measurable.

[HL2] $l(t,\cdot,\cdot)$ is sequentially lower semicontinuous on $X \times Y$ for almost all $t \in J$.

[HL3] $l(t,x,\cdot)$ is convex on Y for each $x \in X$ and almost all $t \in J$.

[HL4] There exist constants $d \ge 0$, $e > 0$, φ is nonnegative and $\varphi \in L^1(J,R)$ such that

$$l(t,x,u) \ge \varphi(t) + d\|x\| + e\|u\|_Y^p.$$

We consider the Lagrange problem (P):

Find $(x^0, u^0) \in X \times U_{ad}$ such that

$$J(x^0, u^0) \leq J(x^u, u), \text{ for all } u \in U_{ad},$$

where

$$J(x^u, u) = \int_0^T l(t,\, x^u(t),\, u(t))dt$$

$x^u(\cdot, x^*)$ denotes the mild solution of system (4.1) corresponding to the control $u \in U_{ad}$.

Lemma 4.2. *Suppose that assumptions* [HA] *and* [HC] *hold. Then the operator* $\mathcal{Q} : L^p(J, Y) \longrightarrow C(J, X)$ *with* $p > 1$*, given by*

$$(\mathcal{Q}f)(\cdot) = \int_0^{\cdot} U(\cdot, s)C(s)u(s)ds$$

is strongly continuous.

Proof. Suppose $\{u^n(\cdot)\} \subseteq L^p(J, Y)$ is bounded, we define $\xi_n(t) = (\mathcal{Q}u^n)(t)$, $t \in J$. By virtue of Theorem 4.1, one can verify that for any but fixed $t \in J$, $\|\xi_n(t)\|$ is bounded. At the same time using properties of $U(\cdot, \cdot)$ (see Lemma 2.1), it is not difficult to verify that $\{\xi_n(\cdot)\}$ is compact and equicontinuous. Due to the Ascoli-Arzela Theorem, $\{\xi(\cdot)\}$ is relatively compact in $C(J, X)$. Obviously, $\mathcal{Q}$ is linear and continuous. Hence $\mathcal{Q}$ is a strongly continuous operator (see P597 of [16]). □

Now we can give the existence of optimal controls for problem (P).

Theorem 4.3. *Suppose* X *is a separable reflexive Banach space. If the assumption* [HL] *and the assumptions of Theorem 4.1 holds, then the problem (P) admits at least one optimal pair.*

Proof. If $\inf\{J(x^u, u) \mid u \in U_{ad}\} = +\infty$, there is nothing to prove.

Assume that $\inf\{J(x^u, u) \mid u \in U_{ad}\} = m < +\infty$. Using assumption [HL], we have $m > -\infty$. By definition of the infimum there exists a minimizing sequence feasible pair $\{(x^n, u^n)\} \subset A_{ad} \equiv \{(x, u) \mid x$ is a mild solution of system (4.1) corresponding to $u \in U_{ad}\}$, such that $J(x^n, u^n) \longrightarrow m$ as $n \longrightarrow +\infty$. Since $\{u_n\} \subseteq U_{ad}$, $\{u_n\}$ is bounded in $L^p(J, Y)$, there exists a subsequence, relabeled as $\{u^n\}$, and $u^0 \in L^p(J, Y)$ such that

$$u^n \xrightarrow{w} u^0 \text{ in } L^p(J, Y)$$

U_{ad} is closed and convex, thanks to the Marzur Lemma, $u^0 \in U_{ad}$.

Suppose x^n is the mild solution of system (4.1) corresponding to u^n ($n = 0, 1, 2, \cdots$), x^n satisfies the following integral equation

$$x^n(t) = U(t,0)[x_0 + g(x^n)] + \\ + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} s^n U(t,s) f(s, x^n(s), (Kx^n)(s), (Hx^n)(s)) ds + \\ + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} s^n U(t,s) C(s) u^n(s) ds.$$

Let $f_n(\theta) \equiv f(\theta, x^n(\theta), (Kx^n)(\theta), (Hx^n)(\theta))$, by assumption [Hf′], we obtain that f_n is a bounded continuous operator from in J into X, hence $f_n(\cdot) \in L^p(J,X)$. Furthermore, $\{f_n(\cdot)\} \subseteq X$, $\{f_n(\cdot)\}$ is bounded in $L^p(J,X)$, there exists a subsequence, relabeled as $\{f_n(\cdot)\}$, and $\widehat{f}(\cdot) \in L^p(J,X)$ such that

$$f_n(\cdot) \xrightarrow{w} \widehat{f}(\cdot) \text{ in } L^p(J,X).$$

By Lemma 4.2, we have

$$\mathcal{Q} f_n \xrightarrow{s} \mathcal{Q}\widehat{f} \text{ in } C(J,X).$$

We consider the following system

$$\begin{cases} D^q x(t) = A(t)x(t) + t^n \widehat{f}(t) + C(t)u^0(t),\ t \in J,\ u \in U_{ad},\ n \in Z^+,\ q \in (0,1), \\ x(0) = g(x) + x_0. \end{cases} \tag{4.2}$$

By Theorem 4.1, we know that system (4.2) has a mild solution

$$\widehat{x}(t) = U(t,0)[x_0 + g(\widehat{x})] + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} s^n U(t,s) \widehat{f}(s)\, ds + \\ + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} s^n U(t,s) C(s) u^0(s) ds.$$

Define

$$\eta_n(t) = \left\| \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} s^n U(t,s) \left[\left(f_n(s) - \widehat{f}(s) \right) + C(s) \left(u^n(s) - u^0(s) \right) \right] ds \right\|,$$

then

$$\eta_n(t) \le \frac{T^{n+q}}{(n+1)\Gamma(q)} \int_0^t \left\| U(t,s) \left[\left(f_n(s) - \widehat{f}(s) \right) + C(s) \left(u^n(s) - u^0(s) \right) \right] \right\| ds.$$

Using Lemma 4.2 again, we have

$$\eta_n \longrightarrow 0 \text{ in } C(J,R) \text{ as } n \longrightarrow \infty.$$

It comes from

$$\|x^n(t)-\widehat{x}(t)\| \le \|U(t,0)[g(x^n)-g(\widehat{x})]\| +$$

$$+\left\| \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} s^n U(t,s) \left[\left(f_n(s) - \widehat{f}(s) \right) + C(s)\left(u^n(s)-u^0(s)\right) \right] ds \right\| \le$$

$$\le Ml_g \|x^n - \widehat{x}\|_C + \eta_n$$

and $Ml_g < 1$, one has

$$0 \le (1-Ml_g)\|x^n-\widehat{x}\|_C \quad \le \quad \eta_n.$$

Then we obtain

$$x^n \longrightarrow \widehat{x} \text{ in } C(J,X) \text{ as } n \longrightarrow \infty.$$

Furthermore, using assumptions [Hf′], [Hk′] and [Hh′], we also obtain

$$f_n(\cdot) \to f(\cdot, \widehat{x}(\cdot), (K\widehat{x})(\cdot), (H\widehat{x})(\cdot)) \text{ in } C(J,X) \text{ as } n \longrightarrow \infty.$$

Using the uniqueness of limit, we have

$$\widehat{f}(t) = f(t, \widehat{x}, (K\widehat{x})(t), (H\widehat{x})(t)).$$

Thus, $\widehat{x}$ can be given by

$$\widehat{x}(t) = U(t,0)[x_0 + g(\widehat{x})] + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} s^n U(t,s) f(s, \widehat{x}, (K\widehat{x})(s), (H\widehat{x})(s)) ds +$$

$$+ \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} s^n U(t,s) C(s) u^0(s) ds,$$

is just a mild solution of system (4.1) corresponding to u^0.

Since $C(J,X) \hookrightarrow L^1(J,X)$, using the assumption [HL] and Balder's theorem, we can obtain

$$m = \lim_{n\to\infty} \int_0^T l\left(t, x^n(t), u^n(t)\right) dt \ge \int_0^T l\left(t, \widehat{x}(t), u^0(t)\right) dt = J\left(\widehat{x}, u^0\right) \ge m.$$

This shows that J attains its minimum at $u^0 \in U_{ad}$. □

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