TREES WITH EQUAL GLOBAL OFFENSIVE k-ALLIANCE AND k-DOMINATION NUMBERS

Mustapha Chellali

Abstract. Let $k \geq 1$ be an integer. A set S of vertices of a graph $G = (V(G), E(G))$ is called a global offensive k-alliance if $|N(v) \cap S| \geq |N(v) - S| + k$ for every $v \in V(G) - S$, where $N(v)$ is the neighborhood of v. The subset S is a k-dominating set of G if every vertex in $V(G) - S$ has at least k neighbors in S. The global offensive k-alliance number $\gamma_o^k(G)$ is the minimum cardinality of a global offensive k-alliance in G and the k-domination number $\gamma_k(G)$ is the minimum cardinality of a k-dominating set of G. For every integer $k \geq 1$ every graph G satisfies $\gamma_o^k(G) \geq \gamma_k(G)$. In this paper we provide for $k \geq 2$ a characterization of trees T with equal $\gamma_o^k(T)$ and $\gamma_k(T)$.

Keywords: global offensive k-alliance number, k-domination number, trees.

Mathematics Subject Classification: 05C69.

1. INTRODUCTION

We begin with some terminology. For a vertex v of a simple graph $G = (V(G), E(G))$, the *open neighborhood* of $v \in V(G)$ is $N(v) = N_G(v) = \{u \in V(G) \mid uv \in E(G)\}$ and the *degree* of v, denoted by $\deg_G(v)$, is $|N_G(v)|$. By $n(G)$ and $\Delta(G) = \Delta$ we denote the *order* and the *maximum degree* of the graph G, respectively. Specifically, for a vertex v in a rooted tree T, we denote by $C(v)$ and $D(v)$ the set of *children* and *descendants*, respectively, of v, and we define $D[v] = D(v) \cup \{v\}$. The *maximal subtree* at v is the subtree of T induced by $D[v]$, and is denoted by T_v.

In [9] Kristiansen, Hedetniemi, and Hedetniemi introduced several types of alliances in graphs, including defensive and offensive alliances. We are interested in a generalization of offensive alliances, namely global offensive k-alliances given by Shafique and Dutton [10, 11]. Let $k \geq 1$ be an integer. A set S of vertices of a graph G is called a *global offensive k-alliance* if $|N(v) \cap S| \geq |N(v) - S| + k$ for every $v \in V(G) - S$ for $1 \leq k \leq \Delta$. The *global offensive k-alliance number* $\gamma_o^k(G)$ is the minimum cardinality of a global offensive k-alliance in G. If S is a global offensive

k-alliance of G and $|S| = \gamma_o^k(G)$, then we say that S is a $\gamma_o^k(G)$-set. Note that a global offensive 1-alliance is a global offensive alliance and a global offensive 2-alliance is a global strong offensive alliance. Recently, Fernau, Rodríguez and Sigarreta showed in [5] that the problem of finding optimal global offensive k-alliances is NP-complete, and Chellali, Haynes, Randerath and Volkmann presented in [3] several bounds on the global offensive k-alliance number.

For a positive integer k, a set of vertices D in a graph G is said to be a *k-dominating set* if each vertex of G not in D has at least k neighbors in D. The order of the smallest k-dominating set of G is called the *k-domination number*, and it is denoted by $\gamma_k(G)$. The concept of k-domination was introduced by Fink and Jacobson in [6, 7], and is studied, for example, in [4, 8] and elsewhere.

Clearly, if S is any global offensive k-alliance, then every vertex of $V(G) - S$ has at least k neighbors in S. Thus S is a k-dominating set of G, and hence $\gamma_k(G) \leq \gamma_o^k(G)$.

In this paper, we provide a characterization of trees with equal global offensive k-alliance and k-domination numbers for every integer $k \geq 2$. Note that a characterization of trees T with $\gamma_1(T) = \gamma_o^1(T)$ has been given by Bouzefrane and Chellali [2].

2. MAIN RESULT

We begin by introducing the following trees defined in [1] by Blidia, Chellali and Volkmann. For a positive integer p, a nontrivial tree T is called *$\mathcal{N}_p$-tree* if T contains a vertex, say w, of degree at least $p-1$ and $\deg_T(x) \leq p-1$ for every vertex of $x \in V(T) - \{w\}$. The vertex w will be called the *special vertex* of T. An $\mathcal{N}_p$-tree with special vertex w is called *exact* if $\deg_T(w) = p-1$.

For the purpose of characterizing trees T with $\gamma_k(T) = \gamma_o^k(T)$ for $k \geq 2$ we define the family $\mathcal{F}_k$ of all trees T that can be obtained from a sequence $T_1, T_2, \ldots, T_p$ $(p \geq 1)$ of trees, where T_1 is an $\mathcal{N}_k$-tree with special vertex w of degree at least $k-1$, $T = T_p$, and, if $p \geq 2$, T_{i+1} can be obtained recursively from T_i by one of the operations listed below.

- Operation $\mathcal{O}_1$: Attach an $\mathcal{N}_k$-tree with special vertex x of degree at least $k+1$ by adding an edge from x to any vertex u of T_i with the condition that if u does not belong to a $\gamma_o^k(T_i)$-set D, then $|N_{T_i}(u) \cap D| > |N_{T_i}(u) - D| + k$.
- Operation $\mathcal{O}_2$: Attach an $\mathcal{N}_k$-tree with special vertex x of degree $k-1$ or k by adding an edge from x to a vertex u of T_i that belongs to a $\gamma_o^k(T_i)$-set.
- Operation $\mathcal{O}_3$: Attach an exact $\mathcal{N}_k$-tree with special vertex x and $q \geq 1$ new trees, all vertices of degree at most $k-1$ and join x and a vertex of each new tree by an edge to a vertex z of T_i of degree exactly $k-1$.

The following observations will be useful for the next.

Observation 2.1. *For every graph G and positive integer k, every vertex with degree at most $k-1$ belongs to every $\gamma_o^k(G)$-set and to every $\gamma_k(G)$-set.*

Observation 2.2. *Let $k \geq 2$ be an integer and T a tree obtained from an $\mathcal{N}_k$-tree H with special vertex w by adding an edge between w and a vertex v of a tree T'. Then $\gamma_o^k(T') \leq \gamma_o^k(T) - |V(H)| + 1$ with equality if:*

1) *v belongs to a $\gamma_o^k(T')$-set.*
2) *$\deg_H(w) \geq k+1$ and v satisfies $|N_{T'}(v) \cap D| > |N_{T'}(v) - D| + k$, where D is a $\gamma_o^k(T')$-set such that $v \notin D$.*

Proof. Let Q be a $\gamma_o^k(T)$-set. Then by Observation 2.1, Q contains $V(H) - \{w\}$ and, without loss of generality, $w \notin Q$ (else replace w in Q by v) and hence $v \in Q$. Thus $Q \cap V(T')$ is a global offensive k-alliance of T', and so $\gamma_o^k(T') \leq \gamma_o^k(T) - |V(H)| + 1$. Now let D' be a $\gamma_o^k(T')$-set. If $v \in D'$, then $D' \cup (V(H) - \{w\})$ is a global offensive k-alliance of T'. If $\deg_H(w) \geq k+1$, $v \notin D'$ and v satisfies $|N_{T'}(v) \cap D'| > |N_{T'}(v) - D'| + k$, then $D' \cup (V(H) - \{w\})$ is a global offensive k-alliance of T' too. In both cases $\gamma_o^k(T) \leq \gamma_o^k(T') + |V(H)| - 1$ and the equality follows. □

By using a similar proof we obtain the following

Observation 2.3. *Let $k \geq 2$ be an integer and T a tree obtained from an $\mathcal{N}_k$-tree H with special vertex w by adding an edge between w and a vertex v of a tree T'. Then $\gamma_k(T') \leq \gamma_k(T) - |V(H)| + 1$ with equality if either $\deg_H(w) \geq k$ or v belongs to a $\gamma_k(T')$-set.*

We state a lemma.

Lemma 2.4. *If $k \geq 2$ and $T \in \mathcal{F}_k$, then $\gamma_o^k(T) = \gamma_k(T)$.*

Proof. Assume that $k \geq 2$ and let T be a tree of $\mathcal{F}_k$. Then T is obtained from a sequence $T_1, T_2, \ldots, T_p$ ($p \geq 1$) of trees, where T_1 is an $\mathcal{N}_k$-tree with special vertex w of degree at least $k-1$, $T = T_p$, and, if $p \geq 2$, T_{i+1} can be obtained recursively from T_i by one of the operations defined above. We will use induction on p. If $p = 1$, then $\gamma_o^k(T_1) = \gamma_k(T_1) = n(T_1)$ or $n(T_1) - 1$ depending on whether w has degree $k-1$ or more, respectively.

Assume now that $p \geq 2$ and that the result holds for all trees $T \in \mathcal{F}_k$ that can be constructed from a sequence of length at most $p-1$, and let $T' = T_{p-1}$. By the inductive hypothesis on $T' \in \mathcal{F}_k$ we have $\gamma_o^k(T') = \gamma_k(T')$. Let T be a tree obtained from T' and consider the following cases.

Assume that T is obtained from T' by using Operation $\mathcal{O}_1$ or $\mathcal{O}_2$. Let H be the added $\mathcal{N}_k$-tree. Then by Observations 2.2 and 2.3, $\gamma_o^k(T) = \gamma_o^k(T') + |V(H)| - 1$, $\gamma_k(T) = \gamma_k(T') + |V(H)| - 1$ and hence $\gamma_o^k(T) = \gamma_k(T)$.

Assume now that T is obtained from T' by using operation $\mathcal{O}_3$. Let H be the added $\mathcal{N}_k$-tree with special vertex x and $H_1, H_2, \ldots, H_q$ the q added new trees attached to z of T'. We further assume that t trees among the q new trees are attached to z by vertices of degree exactly $k-1$, and so such vertices would have degree k in T. It can be seen easily that $\gamma_o^k(T) = \gamma_o^k(T') + |V(H)| - 1 + \sum_{i=1}^{q} |V(H_i)| - t$, and $\gamma_k(T) = \gamma_k(T') + |V(H)| - 1 + \sum_{i=1}^{q} |V(H_i)| - t$. Therefore $\gamma_o^k(T) = \gamma_k(T)$. □

We now are ready to give our main result.

Theorem 2.5. *Let $k \geq 2$ be an integer. A tree T satisfies $\gamma_o^k(T) = \gamma_k(T)$ if and only if either $\Delta(T) \leq k-2$ or $T \in \mathcal{F}_k$.*

Proof. If T is a tree with $\Delta(T) \leq k-2$, then by Observation 2.1, $\gamma_o^k(T) = \gamma_k(T) = n(T)$. If $T \in \mathcal{F}_k$, then by Lemma 2.4, $\gamma_o^k(T) = \gamma_k(T)$.

Let us prove the "only if" part. Let $k \geq 2$ be an integer and T a tree with $\gamma_o^k(T) = \gamma_k(T)$. Suppose that $\Delta(T) \geq k-1$ and let $B(T) = \{x \in V(T) : \deg_T(x) \geq k\}$. We use an induction on the size of $B(T)$. If $|B(T)| = 0$ or 1, then T is an (exact) $\mathcal{N}_k$-tree that belongs to $\mathcal{F}_k$. Let $|B(T)| \geq 2$ and assume that every tree T' with $|B(T')| < |B(T)|$ such that $\gamma_o^k(T') = \gamma_k(T')$ is in $\mathcal{F}_k$. Let T be a tree with $\gamma_o^k(T) = \gamma_k(T)$ and S a $\gamma_o^k(T)$-set.

We now root T at a vertex r of maximum eccentricity. Let w be a vertex of degree at least k at maximum distance from r. We further assume that among such vertices w has maximum degree. Clearly since $k \geq 2, w \neq r$ and the subtree induced by $D(w) \cup \{w\}$ is an $\mathcal{N}_k$-tree with special vertex w of degree at least $k-1$. Note that every vertex in $D(w)$ has degree at most $k-1$ and so $D(w)$ is contained in every $\gamma_o^k(T)$-set and every $\gamma_k(T)$-set. Let u be the parent of w in the rooted tree. We consider the following cases.

Case 1. $\deg_T(w) \geq k+2$. Let $T' = T - T_w$. By Observation 2.3, $\gamma_k(T) = \gamma_k(T') + |V(T_w)| - 1$ and by Observation 2.2, $\gamma_o^k(T') \leq \gamma_o^k(T) - |V(T_w)| + 1$. If $\gamma_o^k(T') < \gamma_o^k(T) - |V(T_w)| + 1$, then using the fact $\gamma_o^k(T) = \gamma_k(T)$ we arrive to $\gamma_o^k(T') < \gamma_k(T')$, a contradiction. Therefore $\gamma_o^k(T') = \gamma_o^k(T) - |V(T_w)| + 1$. Hence we may assume that $w \notin S$ (else replace w by u) and so $S' = S \cap V(T')$ is a $\gamma_o^k(T')$-set. Observe that if $u \notin S'$, then since $w \notin S$ the set S' is a $\gamma_o^k(T')$-set for which u satisfies $|N_{T'}(u) \cap S'| > |N_{T'}(u) - S'| + k$. Now it follows by the previous equalities that $\gamma_o^k(T') = \gamma_k(T')$. If $B(T') = \emptyset$, then $\deg_T(u) = k$ and T' is an exact $\mathcal{N}_k$-tree with special vertex u, that is $T' \in \mathcal{F}_k$. If $B(T') \neq \emptyset$, then clearly $|B(T')| < |B(T)|$ and hence by induction on T', we have $T' \in \mathcal{F}_k$. Therefore in both cases $T \in \mathcal{F}_k$ and is obtained from T' by using Operation $\mathcal{O}_1$.

Case 2. $\deg_T(w) = k+1$. Let $T' = T - T_w$. By Observation 2.3, $\gamma_k(T) = \gamma_k(T') + |V(T_w)| - 1$ and by Observation 2.2, $\gamma_o^k(T') \leq \gamma_o^k(T) - |V(T_w)| + 1$. By using the same argument as that used in Case 1, we obtain $\gamma_o^k(T') = \gamma_o^k(T) - |V(T_w)| + 1$. Also $w \notin S$ (else replace w by u in S) and hence $u \in S$, implying that $S' = S \cap V(T')$ is a $\gamma_o^k(T')$-set, where $u \in S'$. The previous equalities imply that $\gamma_o^k(T') = \gamma_k(T')$. Clearly $|B(T')| < |B(T)|$ but we note that $B(T') \neq \{u\}$ for otherwise $S' - \{u\}$ would be a global offensive k-alliance of T'. Now by induction on T' we have $T' \in \mathcal{F}_k$. Hence $T \in \mathcal{F}_k$ and is obtained from T' by using Operation $\mathcal{O}_2$.

Case 3. $\deg_T(w) = k$. By our choice of w every vertex in $C(u)$ has degree at most k. Recall that $|B(T)| \geq 2$. If $\deg_T(u) \leq k$, then let $T' = T - T_w$. It can be seen that $\gamma_k(T) = \gamma_k(T') - |V(T_w)| + 1$ and $\gamma_o^k(T') = \gamma_o^k(T) - |V(T_w)| + 1$. Therefore $\gamma_o^k(T') = \gamma_k(T')$ and by induction on T' we have $T' \in \mathcal{F}_k$. Since $\deg_{T'}(u) \leq k-1$, u belongs to every $\gamma_o^k(T')$-set. Thus $T \in \mathcal{F}_k$ and is obtained from T' by using Operation $\mathcal{O}_2$. Now assume that $\deg_T(u) = q \geq k+1$, then let $w = w_1, w_2, \ldots, w_{q-k+1}$ be any vertices of $C(u)$, where the first t ($t \geq 1$) vertices have degree exactly k and the remaining vertices have degree at most $k-1$. Let $T' = T - \bigcup_{j=1}^{q+1-k} T_{w_j}$. Note that $\deg_{T'}(u) = k-1$. By Observations 2.1, 2.2 and 2.3, it can be seen easily that

$$\gamma_o^k(T) = \gamma_o^k(T') + \left| \bigcup_{j=1}^{q+1-k} D[w_j] \right| - t,$$

and

$$\gamma_k(T) = \gamma_k(T') + \left| \bigcup_{j=1}^{q+1-k} D[w_j] \right| - t.$$

Therefore $\gamma_o^k(T') = \gamma_k(T')$. Now since $|B(T')| < |B(T)|$ we obtain by induction $T' \in \mathcal{F}_k$. Hence $T \in \mathcal{F}_k$ and is obtained from T' by using Operation $\mathcal{O}_3$. □

REFERENCES

[1] M. Blidia, M. Chellali, L. Volkmann, *Some bounds on the p-domination number in trees*, Discrete Math. **306** (2006), 2031–2037.

[2] M. Bouzefrane, M. Chellali, *On the global offensive alliance number of a tree*, Opuscula Math. **29** (2009), 223–228.

[3] M. Chellali, T.W. Haynes, B. Randerath, L. Volkmann, *Bounds on the global offensive k-alliance number in graphs*, Discuss. Math. Graph Theory **29** (2009), 597–613.

[4] O. Favaron, A. Hansberg, L. Volkmann, *On k-domination and minimum degree in graphs*, J. Graph Theory **57** (2008), 33–40.

[5] H. Fernau, J.A. Rodríguez, J.M. Sigarreta, *Offensive r-alliance in graphs*, Discrete Appl. Math. **157** (2009), 177–182.

[6] J.F. Fink, M.S. Jacobson, *n-domination in graphs*, Graph Theory with Applications to Algorithms and Computer Science, John Wiley and Sons, New York, 1985, 282–300.

[7] J.F. Fink, M.S. Jacobson, *On n-domination, n-dependence and forbidden subgraphs*, Graph Theory with Applications to Algorithms and Computer Science, John Wiley and Sons, New York 1985, 301–311.

[8] A. Hansberg, L. Volkmann, *Lower bounds on the p-domination number in terms of cycles and matching number*, J. Combin. Math. Combin. Comput. **68** (2009), 245–255.

[9] P. Kristiansen, S.M. Hedetniemi, S.T. Hedetniemi, *Alliances in graphs*, J. Combin. Math. Combin. Comput. **48** (2004), 157–177.

[10] K.H. Shafique, R.D. Dutton, *Maximum alliance-free and minimum alliance-cover sets*, Congr. Numer. **162** (2003), 139–146.

[11] K.H. Shafique, R. Dutton, *A tight bound on the cardinalities of maximum alliance-free and minimum alliance-cover sets*, J. Combin. Math. Combin. Comput. **56** (2006), 139–145.

Mustapha Chellali
m_chellali@yahoo.com

University of Blida
LAMDA-RO Laboratory, Department of Mathematics
B.P. 270, Blida, Algeria

Received: August 3, 2009.
Revised: January 15, 2010.
Accepted: January 16, 2010.