DECOMPOSITION OF COMPLETE GRAPHS INTO SMALL GRAPHS

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Abstract. In 1967, A. Rosa proved that if a bipartite graph G with n edges has an α-labeling, then for any positive integer p the complete graph K_{2np+1} can be cyclically decomposed into copies of G. This has become a part of graph theory folklore since then. In this note we prove a generalization of this result. We show that every bipartite graph H which decomposes K_k and K_m also decomposes K_{km}.

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Let G be a graph with at most n vertices. We say that the complete graph K_n has a *G-decomposition* (or that it is *G-decomposable*) if there are subgraphs $G_0, G_1, G_2, \ldots, G_s$ of K_n, all isomorphic to G, such that each edge of K_n belongs to exactly one G_i.

In 1967 A. Rosa [5] introduced some important types of vertex labelings. Graceful labeling (called β-valuation by AR) and rosy labeling (called ρ-valuation by AR) are useful tools for decompositions of complete graphs K_{2n+1} into graphs with n edges. A *labeling* of a graph G with n edges is an injection ρ from $V(G)$, the vertex set of G, into a subset S of the set $\{0, 1, 2, \ldots, 2n\}$ of elements of the additive group Z_{2n+1}. The *length* of an edge $e = xy$ with endvertices x and y is defined as $\ell(xy) = \min\{\rho(x) - \rho(y), \rho(y) - \rho(x)\}$. Notice that the subtraction is performed in Z_{2n+1} and hence $1 \leq \ell(e) \leq n$. If the set of all lengths of the n edges is equal to $\{1, 2, \ldots, n\}$, then ρ is a *rosy labeling*; if moreover $S \subseteq \{0, 1, \ldots, n\}$, then ρ is a *graceful labeling.* A graceful labeling α is said to be an *α-labeling* if there exists a number α_0 with the property that for every edge e in G with endvertices x and y and with $\alpha(x) < \alpha(y)$ it holds that $\alpha(x) \leq \alpha_0 < \alpha(y)$. Obviously, G must be bipartite to allow an α-labeling. For an exhaustive survey of graph labelings, see [3] by J. Gallian.

A. Rosa observed that if a graph G with n edges has a graceful or rosy labeling, then K_{2n+1} can be cyclically decomposed into $2n + 1$ copies of G. It is so because K_{2n+1} has exactly $2n + 1$ edges of length i for every $i = 1, 2, \ldots, n$ and each copy of G contains exactly one edge of each length. The cyclic decomposition is constructed by taking a labeled copy of G, say G_0, and then adding a non-zero element $i \in Z_{2n+1}$ to the label of each vertex of G_0 to obtain a copy G_i for $i = 1, 2, \ldots, 2n$.

If G with n edges has an α-labeling, then we can take p copies of G, say G_0, G_1, G_{p-1}, and label them such that G_0 has the original labels induced by the α-labeling, and for every $i = 1, 2, \ldots, p-1$ the vertices with lower labels (that is, with $\alpha(x) \leq \alpha_0$) will keep their labels, while the vertices with high labels will increase their labels by in. This way a copy G_i contains edges of lengths $in+1, in+2, \ldots, (i+1)n$. Therefore all p copies together contain np edges of lengths $1, 2, \ldots, np$. It follows that the graph consisting of these p edge-disjoint copies of G decomposes cyclically the complete graph K_{2np+1} and consequently, G itself decomposes K_{2np+1}.

We summarize these classical Rosa's results in the following theorem.

Theorem 1. *Let G be a graph with n edges. If G allows a rosy labeling, then it decomposes K_{2n+1}, if G allows an α-labeling, then it decomposes K_{2np+1} for every $p > 0$.*

To guarantee a G-decomposition of K_{2np+1}, the condition of the existence of an α-labeling can be relaxed. S. El-Zanati, C. Vanden Eynden, and N. Punnim [2] defined a *ρ^+-labeling* of a bipartite graph G with bipartition X, Y as a rosy labeling with the additional property that for every edge $xy \in E(G)$ with $x \in X, y \in Y$ it holds that $\rho^+(x) < \rho^+(y)$. Their theorem then follows by arguments similar to those for the α-labeling.

Theorem 2. *If a bipartite graph G with n edges has a ρ^+-labeling, then there exists a cyclic G-decomposition of K_{2np+1} for any positive integer p.*

In [1] M. Buratti and A. Pasotti proved a result on difference matrices, which is in [4] restated as follows.

Theorem 3. *If a graph G with n edges and chromatic number $\chi(G)$ cyclically decomposes K_k and K_m, where $k \equiv m \equiv 1 \pmod{2n}$ and $\chi(G)$ does not exceed the smallest prime factor of m, then there exists a cyclic G-decomposition of K_{km}.*

Because a bipartite graph has $\chi(G) = 2$, the following corollary is easy to prove. It was stated in [1] in a more general form related to Theorem 3.

Corollary 4. *If a bipartite graph G with n edges has a ρ-labeling, then there exists a cyclic G-decomposition of $K_{(2n+1)^r}$ for any positive integer r.*

Our goal is to show that if we restrict ourselves to bipartite graphs while assuming the existence of *any* G-decomposition rather than a cyclic one, we can still get a result similar to Theorem 3.

First we prove a related useful result for decompositions of complete multipartite graphs. We recall that a *composition* $G[H]$ of graphs G and H (also called a *lexicographic product*) is a graph that arises from G by replacing each vertex of G by a copy of H and each edge of G by $K_{m,m}$, where m is the order of H. In particular, if $H = \overline{K}_m$, the graph consisting of m isolated vertices, then we say that we *blow up* G into $G[\overline{K}_m]$.

Observation 5. If a bipartite graph G decomposes K_k, then G also decomposes the complete k-partite graph $K_{m,m,\ldots,m}$ for any $m \geq 2$.

Proof. Because $K_{m,m,\ldots,m} = K_k[\overline{K}_m]$, it is obvious that it can be decomposed into $G[\overline{K}_m]$ by blowing up K_k and concurrently every copy of G. Moreover, since G is bipartite, $G[\overline{K}_m]$ is also bipartite. Let X, Y be the partite sets of G and $\overline{X}, \overline{Y}$ the corresponding bipartition of $G[\overline{K}_m]$. We need to decompose $G[\overline{K}_m]$ into m^2 edge-disjoint copies of G.

We label the vertices of K_k by the elements of Z_k and the vertices of $K_{m,m,\ldots,m} = K_k[\overline{K}_m]$ by the elements (i, j) of the group $Z_k \times Z_m$.

Now we construct m^2 copies of G, denoted by G_{ij} for $i, j = 0, 1, \ldots, m-1$. If xy is an edge in G with $x \in X, y \in Y$ (we here identify vertices with their labels, so in fact x and y are the *labels* of these vertices), then in G_{ij} there will be the edge $(x, i)(y, j)$. Therefore, every G_{ij} contains an edge $(x, i)(y, j)$ if and only if G contains the edge xy. On the other hand, for every complete bipartite graph $K^{xy}_{m,m}$ in $G[\overline{K}_m]$ corresponding to an edge $xy \in G$ we have each of its m^2 edges in precisely one of the graphs G_{ij}.

It should be clear that this way $G[\overline{K}_m]$ is decomposed into m^2 copies of G. Because at the same time $G[\overline{K}_m]$ decomposes $K_{m,m,\ldots,m}$, it is obvious that G decomposes $K_{m,m,\ldots,m}$. □

Now it is easy to observe that the following is true.

Theorem 6. *If a bipartite graph G decomposes K_k and K_m, then G also decomposes the complete graph K_{km}.*

The following equivalent of Corollary 4 is now obvious.

Theorem 7. *Let G be a bipartite graph which decomposes K_s. Then G decomposes also K_{s^r} for any $r \geq 1$.*

Proof. We prove the claim by induction on r. For $r = 1$ we get our assumption that G decomposes K_s. Now assume that G decomposes $K_{s^{r-1}}$. First decompose K_{s^r} into s copies of $K_{s^{r-1}}$ and a complete s-partite graph B with partite sets of size s^{r-1}. By induction hypothesis, each $K_{s^{r-1}}$ can be decomposed into G. The existence of a decomposition of the complete s-partite graph B follows from our assumption that G decomposes K_s and from Observation 5, where we set $k = s$ and $m = s^{r-1}$. Therefore, G decomposes K_{s^r}. □

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