CONVERGENCE OF AN IMPLICIT ITERATION PROCESS FOR A FINITE FAMILY OF ASYMPTOTICALLY QUASI-NONEXPANSIVE MAPPINGS IN CONVEX METRIC SPACES

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Abstract. In this paper, we give some necessary and sufficient conditions for an implicit iteration process with errors for a finite family of asymptotically quasi-nonexpansive mappings converging to a common fixed of the mappings in convex metric spaces. Our results extend and improve some recent results of Sun, Wittmann, Xu and Ori, and Zhou and Chang.

Keywords: implicit iteration process, finite family of asymptotically quasi-nonexpansive mappings, common fixed point, convex metric space.

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1. INTRODUCTION AND PRELIMINARIES

Throughout this paper, we assume that X is a metric space and let $F(T_i) = \{x \in X : T_i x = x\}$ be the set of all fixed points of the mappings T_i $(i = 1, 2, \dots, N)$ respectively. The set of common fixed points of T_i $(i = 1, 2, \dots, N)$ denoted by $\mathcal{F}$, that is, $\mathcal{F} = \bigcap_{i=1}^{N} F(T_i)$.

Definition 1.1 ([4, 5])**.** Let $T\colon X \to X$ be a mapping.

(1) The mapping T is said to be nonexpansive if

$$d(Tx, Ty) \le d(x, y), \quad \forall x, y \in D(T).$$

(2) The mapping T is said to be quasi-nonexpansive if $F(T) \neq \emptyset$ and

$$d(Tx, p) \le d(x, p), \quad \forall x \in D(T),\ \forall p \in F(T).$$

(3) The mapping T is said to be asymptotically nonexpansive if there exists a sequence $k_n \in [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$d(T^n x, T^n y) \leq k_n d(x, y), \quad \forall x, y \in D(T), \quad \forall n \in \mathbb{N}.$$

(4) The mapping T is said to be asymptotically quasi-nonexpansive if $F(T) \neq \emptyset$ and there exists a sequence $k_n \in [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$d(T^n x, p) \leq k_n d(x, p), \quad \forall x \in D(T),\ \forall p \in F(T),\ \forall n \in \mathbb{N}.$$

Remark 1.2. (i) From the definition 1.1, it follows that if $F(T)$ is nonempty, then a nonexpansive mapping is quasi-nonexpansive, and an asymptotically nonexpansive mapping is asymptotically quasi-nonexpansive. But the converse does not hold.

(ii) It is obvious that if T is nonexpansive, then it is asymptotically nonexpansive with the constant sequence $\{1\}$.

In 2001, Xu and Ori [16] have introduced an implicit iteration process for a finite family of nonexpansive mappings in a Hilbert space H. Let C be a nonempty subset of H. Let $T_1, T_2, \ldots, T_N$ be self mappings of C and suppose that $\mathcal{F} = \bigcap_{i=1}^N F(T_i) \neq \emptyset$, the set of common fixed points of $T_i, i = 1, 2, \ldots, N$. An implicit iteration process for a finite family of nonexpansive mappings is defined as follows: Let $\{t_n\}$ a real sequence in $(0, 1)$, $x_0 \in C$:

$$\begin{aligned}
x_1 &= t_1 x_0 + (1 - t_1) T_1 x_1, \\
x_2 &= t_2 x_1 + (1 - t_2) T_2 x_2, \\
\ldots &= \ldots \\
x_N &= t_N x_{N-1} + (1 - t_N) T_N x_N, \\
x_{N+1} &= t_{N+1} x_N + (1 - t_{N+1}) T_1 x_{N+1}, \\
\ldots &= \ldots
\end{aligned}$$

which can be written in the following compact form:

$$x_n = t_n x_{n-1} + (1 - t_n) T_n x_n, \quad n \geq 1, \tag{1.1}$$

where $T_k = T_{k(modN)}$. (Here the mod N function takes values in the set $\{1, 2, \ldots, N\}$.)

In 2003, Sun [12] extend the process (1.1) to a process for a finite family of asymptotically quasi-nonexpansive mappings, with $\{\alpha_n\}$ a real sequence in $(0, 1)$ and an initial point $x_0 \in C$, which is defined as follows:

$$x_1 = \alpha_1 x_0 + (1 - \alpha_1) T_1 x_1,$$

$$
\begin{aligned}
x_2 &= \alpha_2 x_1 + (1-\alpha_2) T_2 x_2,\\
\ldots &= \ldots\\
x_N &= \alpha_N x_{N-1} + (1-\alpha_N) T_N x_N,\\
x_{N+1} &= \alpha_{N+1} x_N + (1-\alpha_{N+1}) T_1^2 x_{N+1},\\
\ldots &= \ldots\\
x_{2N} &= \alpha_{2N} x_{2N-1} + (1-\alpha_{2N}) T_N^2 x_{2N},\\
x_{2N+1} &= \alpha_{2N+1} x_{2N} + (1-\alpha_{2N+1}) T_1^3 x_{2N+1},\\
\ldots &= \ldots
\end{aligned}
$$

which can be written in the following compact form:

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n) T_i^k x_n,\ n \geq 1, \tag{1.2}$$

where $n = (k-1)N + i, \quad i \in \{1, 2, \ldots, N\}$.

Sun [12] proved the strong convergence of the process (1.2) to a common fixed point in real uniformly convex Banach spaces, requiring only one member T in the family $\{T_i : i = 1, 2, \ldots, N\}$ to be semi compact. The result of Sun [12] generalized and extended the corresponding main results of Wittmann [15] and Xu and Ori [16].

The purpose of this paper is to study the convergence of an implicit iteration process with errors for a finite family of asymptotically quasi-nonexpansive mappings in convex metric spaces. The results presented in this paper extend and improve the corresponding results of Sun [12], Wittmann [15], Xu and Ori [16] and Zhou and Chang [17] and many others.

For the sake of convenience, we also recall some definitions and notations.

In 1970, Takahashi [13] introduced the concept of convexity in a metric space and the properties of the space.

Definition 1.3 ([13])**.** Let (X, D) be a metric space and $I = [0, 1]$. A mapping $W \colon X \times X \times I \to X$ is said to be a convex structure on X if for each $(x, y, \lambda) \in X \times X \times I$ and $u \in X$,

$$d(u, W(x, y, \lambda)) \leq \lambda d(u, x) + (1-\lambda) d(u, y).$$

X together with a convex structure W is called a *convex metric space*, denoted by (X, d, W). A nonempty subset K of X is said to be *convex* if $W(x, y, \lambda) \in K$ for all $(x, y, \lambda) \in K \times K \times I$.

Remark 1.4. Every normed space is a convex metric space, where a convex structure $W(x, y, z; \alpha, \beta, \gamma) = \alpha x + \beta y + \gamma z$, for all $x, y, z \in E$ and $\alpha, \beta, \gamma \in I$ with $\alpha + \beta + \gamma = 1$. In fact,

$$
\begin{aligned}
d(u, W(x, y, z; \alpha, \beta, \gamma)) &= \|u - (\alpha x + \beta y + \gamma z)\| \leq\\
&\leq \alpha \|u - x\| + \beta \|u - y\| + \gamma \|u - z\| =\\
&= \alpha d(u, x) + \beta d(u, y) + \gamma d(u, z), \quad \forall u \in X.
\end{aligned}
$$

But there exists some convex metric spaces which can not be embedded into a normed space.

Example 1.5. Let $X = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 > 0, x_2 > 0, x_3 > 0\}$. For $x = (x_1, x_2, x_3), y = (y_1, y_2, y_3) \in X$ and $\alpha, \beta, \gamma \in I$ with $\alpha + \beta + \gamma = 1$, we define a mapping $W\colon X^3 \times I^3 \to X$ by

$$W(x, y, z; \alpha, \beta, \gamma) = (\alpha x_1 + \beta y_1 + \gamma z_1, \alpha x_2 + \beta y_2 + \gamma z_2, \alpha x_3 + \beta y_3 + \gamma z_3)$$

and define a metric $d\colon X \times X \to [0, \infty)$ by

$$d(x, y) = |x_1 y_1 + x_2 y_2 + x_3 y_3|.$$

Then we can show that (X, d, W) is a convex metric space, but it is not a normed space.

Example 1.6. Let $Y = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 > 0, x_2 > 0\}$. For each $x = (x_1, x_2), y = (y_1, y_2) \in Y$ and $\lambda \in I$. We define a mapping $W\colon Y^2 \times I \to Y$ by

$$W(x, y; \lambda) = \left(\lambda x_1 + (1-\lambda) y_1, \frac{\lambda x_1 x_2 + (1-\lambda) y_1 y_2}{\lambda x_1 + (1-\lambda) y_1}\right)$$

and define a metric $d\colon Y \times Y \to [0, \infty)$ by

$$d(x, y) = |x_1 - y_1| + |x_1 x_2 - y_1 y_2|.$$

Then we can show that (Y, d, W) is a convex metric space, but it is not a normed space.

Definition 1.7. Let (X, d, W) be a convex metric space with a convex structure W and let $T_1, T_2, \ldots, T_N\colon X \to X$ be N asymptotically quasi-nonexpansive mappings. For any given $x_0 \in X$, the iteration process $\{x_n\}$ defined by

$$\begin{aligned}
x_1 &= W(x_0, T_1 x_1, u_1; \alpha_1, \beta_1, \gamma_1),\\
\ldots &= \ldots\\
x_N &= W(x_{N-1}, T_N x_N, u_N; \alpha_N, \beta_N, \gamma_N),\\
x_{N+1} &= W(x_N, T_1^2 x_{N+1}, u_{N+1}; \alpha_{N+1}, \beta_{N+1}, \gamma_{N+1}),\\
\ldots &= \ldots\\
x_{2N} &= W(x_{2N-1}, T_N^2 x_{2N}, u_{2N}; \alpha_{2N}, \beta_{2N}, \gamma_{2N}),\\
x_{2N+1} &= W(x_{2N}, T_1^3 x_{2N+1}, u_{2N+1}; \alpha_{2N+1}, \beta_{2N+1}, \gamma_{2N+1}),\\
\ldots &= \ldots
\end{aligned}$$

which can be written in the following compact form:

$$x_n = W(x_{n-1}, T^n_{n(modN)} x_n, u_n; \alpha_n, \beta_n, \gamma_n), \quad n \geq 1, \tag{1.3}$$

where $\{u_n\}$ is a bounded sequence in X, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are three sequences in $[0, 1]$ such that $\alpha_n + \beta_n + \gamma_n = 1$ for $n = 1, 2, \ldots$. Iteration process (1.3) is called the implicit iteration process with errors for a finite family of mappings T_i $(i = 1, 2, \ldots, N)$.

If $u_n = 0$ in (1.3) then,

$$x_n = W(x_{n-1}, T^n_{n(modN)}x_n; \alpha_n, \beta_n), \quad n \geq 1, \tag{1.4}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ are two sequences in $[0,1]$ such that $\alpha_n + \beta_n = 1$ for $n = 1, 2, \ldots$. Iteration process (1.4) is called the implicit iteration process for a finite family of mappings T_i $(i = 1, 2, \ldots, N)$.

Proposition 1.8. *Let $T_1, T_2, \ldots, T_N \colon X \to X$ be N asymptotically nonexpansive mappings. Then there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that*

$$d(T^n_i x, T^n_i y) \leq k_n d(x, y), \quad \forall n \geq 1, \tag{1.5}$$

for all $x, y \in X$ and for each $i = 1, 2, \ldots, N$.

Proof. Since for each $i = 1, 2, \ldots, N$, $T_i \colon X \to X$ is an asymptotically nonexpansive mapping, there exists a sequence $\{k^{(i)}_n\} \subset [1, \infty)$ with $k^{(i)}_n \to 1$ as $n \to \infty$ such that

$$d(T^n_i x, T^n_i y) \leq k^{(i)}_n d(x, y), \quad \forall n \geq 1.$$

Letting

$$k_n = \max\{k^{(1)}_n, k^{(2)}_n, \ldots, k^{(N)}_n\},$$

therefore we have $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ and

$$d(T^n_i x, T^n_i y) \leq k^{(i)}_n d(x, y) \leq k_n d(x, y), \quad \forall n \geq 1,$$

for all $x, y \in X$ and for each $i = 1, 2, \ldots, N$. □

The above theorem is also holds for asymptotically quasi-nonexpansive mappings since an asymptotically nonexpansive mapping with a nonempty fixed point set is called an asymptotically quasi-nonexpansive mapping.

Remark 1.9. We see, from the proof of the preceding proposition, that

$$\sum_{n=1}^{\infty}(k_n - 1) < \infty \Longleftrightarrow \sum_{n=1}^{\infty}(k^{(i)}_n - 1) < \infty,$$

for all $i \in \{1, 2, \ldots, N\}$.

2. MAIN RESULTS

In order to prove our main result of this paper, we need the following lemma.

Lemma 2.1 ([7])**.** *Let $\{a_n\}$, $\{b_n\}$ and $\{r_n\}$ be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leq (1 + r_n)a_n + b_n, \quad n \geq 1.$$

If $\sum_{n=1}^{\infty} b_n < \infty$ and $\sum_{n=1}^{\infty} r_n < \infty$. Then:

(a) $\lim_{n\to\infty} a_n$ *exists.*
(b) *If* $\liminf_{n\to\infty} a_n = 0$, *then* $\lim_{n\to\infty} a_n = 0$.

Now we state and prove our main theorems of this paper.

Theorem 2.2. *Let* (X, d, W) *be a complete convex metric space. Let* $T_1, T_2, \ldots, T_N : X \to X$ *be* N *asymptotically quasi-nonexpansive mappings. Suppose* $\mathcal{F} = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$. *Let* $\{u_n\}$ *be a bounded sequence in* X, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ *be three sequences in* $[0,1]$, $\{\alpha_n\} \subset (s, 1-s)$ *for some* $s \in (0,1)$ *and* $\{k_n\}$ *be the sequence defined by* (1.5) *satisfying the following conditions:*

(i) $\alpha_n + \beta_n + \gamma_n = 1, \quad \forall n \geq 1$;
(ii) $\sum_{n=1}^{\infty}(k_n - 1) < \infty$;
(iii) $\sum_{n=1}^{\infty} \gamma_n < \infty$.

Then the implicit iteration process with errors $\{x_n\}$ *generated by* (1.3) *converges to a common fixed point of* $\{T_1, T_2, \ldots, T_N\}$ *if and only if*

$$\liminf_{n\to\infty} D_d(x_n, \mathcal{F}) = 0,$$

where $D_d(y, \mathcal{F})$ *denotes the distance from* y *to the set* $\mathcal{F}$, *that is,* $D_d(y, \mathcal{F}) = \inf_{z\in\mathcal{F}} d(y, z)$.

Proof. The necessity is obvious. Now, we will only prove the sufficient condition. Setting $k_n = 1+\lambda_n$ with $\lim_{n\to\infty} \lambda_n = 0$. Since $\sum_{n=1}^{\infty}(k_n - 1) < \infty$, so $\sum_{n=1}^{\infty} \lambda_n < \infty$. For any $p \in \mathcal{F}$, from (1.3), it follows that

$$\begin{aligned} d(x_n, p) &= d(W(x_{n-1}, T^n_{n(modN)}x_n, u_n; \alpha_n, \beta_n, \gamma_n), p) \leq \\ &\leq \alpha_n d(x_{n-1}, p) + \beta_n d(T^n_{n(modN)}x_n, p) + \gamma_n d(u_n, p) \leq \\ &\leq \alpha_n d(x_{n-1}, p) + \beta_n(1+\lambda_n)d(x_n, p) + \gamma_n d(u_n, p) \leq \\ &\leq \alpha_n d(x_{n-1}, p) + (1-\alpha_n)(1+\lambda_n)d(x_n, p) + \gamma_n d(u_n, p) \leq \\ &\leq \alpha_n d(x_{n-1}, p) + (1-\alpha_n+\lambda_n)d(x_n, p) + \gamma_n d(u_n, p) \end{aligned} \tag{2.1}$$

for all $p \in \mathcal{F}$.

Therefore we have

$$d(x_n, p) \leq d(x_{n-1}, p) + \frac{\lambda_n}{\alpha_n} d(x_n, p) + \frac{\gamma_n}{\alpha_n} d(u_n, p). \tag{2.2}$$

Since $0 < s < \alpha_n < 1 - s$, it follows from (2.2) that

$$d(x_n, p) \leq d(x_{n-1}, p) + \frac{\lambda_n}{s} d(x_n, p) + \frac{\gamma_n}{s} d(u_n, p). \tag{2.3}$$

Since $\sum_{n=1}^{\infty} \lambda_n < \infty$, there exists a positive integer n_0 such that $s - \lambda_n > 0$ and $\lambda_n < \frac{s}{2}$ and for all $n \geq n_0$.

Thus, we have

$$d(x_n,p) \leq \Big(1+\frac{\lambda_n}{s-\lambda_n}\Big)d(x_{n-1},p)+\frac{\gamma_n}{s-\lambda_n}d(u_n,p). \tag{2.4}$$

It follows from (2.4) that, for each $n=(n-1)N+i \geq n_0$, we have

$$d(x_n,p) \leq \Big(1+\frac{2\lambda_n}{s}\Big)d(x_{n-1},p)+\frac{2\gamma_n}{s}d(u_n,p). \tag{2.5}$$

Setting $b_n=\frac{2\lambda_n}{s}$, where $n=(n-1)N+i, i \in \{1,2,\ldots,N\}$, then we obtain

$$d(x_n,p) \leq (1+b_n)d(x_{n-1},p)+\frac{2M}{s}\gamma_n, \quad \forall p \in \mathcal{F}, \tag{2.6}$$

where, $M=\sup_{n\geq 1} d(u_n,p)$. This implies that

$$D_d(x_n,\mathcal{F}) \leq (1+b_n)d(x_{n-1},\mathcal{F})+\frac{2M}{s}\gamma_n. \tag{2.7}$$

Since $\sum_{n=1}^{\infty}\lambda_n<\infty$, it follows that $\sum_{n=1}^{\infty}b_n<\infty$ and $\sum_{n=1}^{\infty}\gamma_n<\infty$, thus from Lemma 2.1, we have

$$\lim_{n\to\infty} D_d(x_n,\mathcal{F})=0.$$

Next, we will prove that $\{x_n\}$ is a Cauchy sequence. Note that when $x>0$, $1+x\leq e^x$, from (2.6) we have

$$\begin{aligned}
d(x_{n+m},p) &\leq (1+b_{n+m})d(x_{n+m-1},p)+\frac{2M}{s}\gamma_{n+m} \leq \\
&\leq e^{b_{n+m}}d(x_{n+m-1},p)+\frac{2M}{s}\gamma_{n+m} \leq \\
&\leq e^{b_{n+m}}\Big[e^{b_{n+m-1}}d(x_{n+m-2},p)+\frac{2M}{s}\gamma_{n+m-1}\Big]+\frac{2M}{s}\gamma_{n+m} \leq \\
&\leq e^{(b_{n+m}+b_{n+m-1})}d(x_{n+m-2},p)+\frac{2M}{s}e^{b_{n+m}}[\gamma_{n+m}+\gamma_{n+m-1}] \leq \\
&\leq \ldots \\
&\leq e^{(b_{n+m}+b_{n+m-1}+\ldots+b_{n+1})}d(x_n,p)+ \\
&\quad +\frac{2M}{s}e^{(b_{n+m}+b_{n+m-1}+\ldots+b_{n+2})}[\gamma_{n+m}+\gamma_{n+m-1}+\ldots+\gamma_{n+1}] \leq \\
&\leq e^{\sum_{k=n+1}^{n+m}b_k}d(x_n,p)+\frac{2M}{s}e^{\sum_{k=n+2}^{n+m}b_k}\sum_{j=n+1}^{n+m}\gamma_j \leq \\
&\leq e^{\sum_{k=n+1}^{n+m}b_k}d(x_n,p)+\frac{2M}{s}e^{\sum_{k=n+1}^{n+m}b_k}\sum_{j=n+1}^{n+m}\gamma_j \leq \\
&\leq e^{\sum_{k=n+1}^{n+m}b_k}\Big\{d(x_n,p)+\frac{2M}{s}\sum_{j=n+1}^{n+m}\gamma_j\Big\} \leq \\
&\leq M'\Big\{d(x_n,p)+\frac{2M}{s}\sum_{j=n+1}^{n+m}\gamma_j\Big\}<\infty,
\end{aligned} \tag{2.8}$$

for all $p \in \mathcal{F}$ and $n, m \in \mathbb{N}$, where $M' = e^{\sum_{k=n+1}^{n+m} b_k} < \infty$. Since $\lim_{n\to\infty} D_d(x_n, \mathcal{F}) = 0$ and $\sum_{n=1}^{\infty} \gamma_n < \infty$, there exists a natural number n_1 such that for all $n \geq n_1$,

$$D_d(x_n, \mathcal{F}) < \frac{\varepsilon}{4M'} \quad \text{and} \quad \sum_{j=n_1+1}^{\infty} \gamma_j < \frac{s \cdot \varepsilon}{8MM'}.$$

Thus there exists a point $p_1 \in \mathcal{F}$ such that $d(x_{n_1}, p_1) < \frac{\varepsilon}{4M'}$, by the definition of $D_d(x_n, \mathcal{F})$. It follows from (2.8) that for all $n \geq n_1$ and $m \geq 0$,

$$\begin{aligned} d(x_{n+m}, x_n) &\leq d(x_{n+m}, p_1) + d(x_n, p_1) \leq \\ &\leq M' d(x_{n_1}, p_1) + \frac{2MM'}{s} \sum_{j=n_1+1}^{n+m} \gamma_j + M' d(x_{n_1}, p_1) + \frac{2MM'}{s} \sum_{j=n_1+1}^{n+m} \gamma_j < \\ &< M' \cdot \frac{\varepsilon}{4M'} + \frac{2MM'}{s} \cdot \frac{s \cdot \varepsilon}{8MM'} + M' \cdot \frac{\varepsilon}{4M'} + \frac{2MM'}{s} \cdot \frac{s \cdot \varepsilon}{8MM'} < \\ &< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon. \end{aligned}$$

This implies that $\{x_n\}$ is a Cauchy sequence. Since the space is complete, the sequence $\{x_n\}$ is convergent. Let $\lim_{n\to\infty} x_n = p$. Moreover, since the set of fixed points of an asymptotically quasi-nonexpansive mapping is closed, so is $\mathcal{F}$, thus $p \in \mathcal{F}$ from $\lim_{n\to\infty} D_d(x_n, \mathcal{F}) = 0$, that is, p is a common fixed point of the mappings $\{T_1, T_2, \ldots, T_N\}$. This completes the proof. □

If $u_n = 0$, in Theorem 2.2, we can easily obtain the following theorem.

Theorem 2.3. *Let (X, d, W) be a complete convex metric space. Let T_1, $T_2, \ldots, T_N \colon X \to X$ be N asymptotically quasi-nonexpansive mappings. Suppose $\mathcal{F} = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$. Let $x_0 \in X$ and $\{\alpha_n\}$, $\{\beta_n\}$ be two sequences in $[0, 1]$, $\{k_n\}$ be the sequence defined by (1.5) and $\{\alpha_n\} \subset (s, 1-s)$ for some $s \in (0, 1)$ satisfying the following conditions:*

(i) $\alpha_n + \beta_n = 1, \quad \forall n \geq 1$;
(ii) $\sum_{n=1}^{\infty} (k_n - 1) < \infty$.

Then the implicit iteration process $\{x_n\}$ generated by (1.3) converges to a common fixed point of $\{T_1, T_2, \ldots, T_N\}$ if and only if

$$\liminf_{n\to\infty} D_d(x_n, \mathcal{F}) = 0.$$

From Theorem 2.2, we can easily obtain the following theorem.

Theorem 2.4. *Let (X, d, W) be a complete convex metric space. Let T_1, $T_2, \ldots, T_N \colon X \to X$ be N quasi-nonexpansive mappings. Suppose $\mathcal{F} = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$ and $x_0 \in X$. Let $\{u_n\}$ be an arbitrary bounded sequence in X, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ be three sequences in $[0, 1]$ satisfying the following conditions:*

(i) $\alpha_n + \beta_n + \gamma_n = 1, \quad \forall n \geq 1$;
(ii) $\{\alpha_n\} \subset (s, 1-s)$ *for some* $s \in (0, 1)$;
(iii) $\sum_{n=1}^{\infty} \gamma_n < \infty$.

Then the implicit iteration process with errors $\{x_n\}$ *generated by* (1.3) *converges to a common fixed point of* $\{T_1, T_2, \ldots, T_N\}$ *if and only if*

$$\liminf_{n\to\infty} D_d(x_n, \mathcal{F}) = 0.$$

Remark 2.5. Our results extend and improve the corresponding results of Wittmann [15] and Xu and Ori [16] to the case of a more general class of nonexpansive mappings and implicit iteration process with errors.

Remark 2.6. Our results also extend and improve the corresponding results of Sun [12] to the case of an implicit iteration process with errors.

Remark 2.7. The main result of this paper is also an extension and improvement of the well-known corresponding results in [1–11].

Remark 2.8. Our results also extend and improve the corresponding results of Zhou and Chang [17] to the case of a more general class of asymptotically nonexpansive mappings.

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