STRONG CONVERGENCE THEOREM OF A HYBRID PROJECTION ALGORITHM FOR A FAMILY OF QUASI-ϕ-ASYMPTOTICALLY NONEXPANSIVE MAPPINGS

J.F. Tang, S.S. Chang, M. Liu, J.A. Liu

Abstract. The main purpose of this paper is by using a new hybrid projection iterative algorithm to prove some strong convergence theorems for a family of quasi-ϕ-asymptotically nonexpansive mappings. The results presented in the paper improve and extend the corresponding results announced by some authors.

Keywords: quasi-ϕ-asymptotically nonexpansive mapping, asymptotically regular mapping, hybrid projection iterative algorithm, strong convergence theorem.

Mathematics Subject Classification: 47H09, 47H10.

1. INTRODUCTION

Throughout this paper, we assume that E is a real Banach space, E^* is the dual space of E, C is a nonempty closed convex subset of E, and $\langle \cdot, \cdot \rangle$ is the pairing between E and E^*. Recall that a mapping $T : C \to C$ is said to be asymptotically nonexpansive [1] if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\| \ \forall\, x, y \in C \quad \text{and} \quad \forall\, n \geq 1. \tag{1.1}$$

In recent years, nonexpansive mappings and asymptotically nonexpansive mappings have been studied extensively by many authors. In 2003, Nakajo and Takahashi [2] proposed the following modification of the Mann iteration method for a nonexpansive mapping T in a Hilbert space H:

$$\begin{cases} x_0 \in C, \\ y_n = \alpha_n x_n + (1 - \alpha_n) T x_n, \\ C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}(x_0). \end{cases} \tag{1.2}$$

where C is a closed convex subset of H, and P_K is the metric projection from H onto a closed convex subset K of H. They proved that if the sequence $\{\alpha_n\}$ is bounded above from one, then the sequence $\{x_n\}$ generated by (1.2) converges strongly to $P_{F(T)}(x_0)$.

In 2006, Kim and Xu [4] proposed the following modification of the Mann iteration method for a asymptotically nonexpansive mapping T in a Hilbert space H:

$$\begin{cases} x_0 \in C, \\ y_n = \alpha_n x_n + (1-\alpha_n)T^n x_n, \\ C_n = \{z \in C : \|y_n - z\|^2 \le \|x_n - z\|^2 + \theta_n\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n \rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}(x_0), \end{cases} \tag{1.3}$$

where C is a bounded closed convex subset and $\theta_n = (1-\alpha_n)(k_n^2 - 1)(diamC)^2 \to 0$ $(n \to \infty)$. They proved that if the sequence $\{\alpha_n\}$ is bounded above from one, then the sequence $\{x_n\}$ generated by (1.3) converges strongly to $P_{F(T)}(x_0)$.

In 2005, Matsushita and Takahashi [3] proposed the following hybrid iteration method with generalized projection for a relatively nonexpansive mapping T in a Banach space E:

$$\begin{cases} x_0 \in C, \\ y_n = J^{-1}(\alpha_n J x_n + (1-\alpha_n) J T x_n), \\ C_n = \{z \in C : \phi(z, y_n) \le \phi(z, x_n)\}, \\ Q_n = \{z \in C : \langle x_n - z, Jx_0 - Jx_n \rangle \ge 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n}(x_0). \end{cases} \tag{1.4}$$

Under suitable conditions they proved that the sequence $\{x_n\}$ generated by (1.4) converges strongly to $\Pi_{F(T)}(x_0)$.

In 2009, Zhou and Gao [5] proposed the following modified hybrid iteration method with generalized projection for a family of closed and quasi-ϕ-asymptotically nonexpansive mappings which are asymptotically regular in a Banach space E:

$$\begin{cases} x_0 \in C, \\ y_{n,i} = J^{-1}(\alpha_n J x_n + (1-\alpha_n) J T_i^n x_n), \\ C_{n,i} = \{z \in C : \phi(z, y_{n,i}) \le \phi(z, x_n) + \xi_{n,i}\}, \\ C_n = \cap_{i \in I} C_{n,i}, \\ Q_0 = C, \\ Q_n = \{z \in Q_{n-1} : \langle x_n - z, Jx_0 - Jx_n \rangle \ge 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n}(x_0). \end{cases} \tag{1.5}$$

Under suitable conditions they proved that the sequence $\{x_n\}$ generated by (1.5) converges strongly to $\Pi_{\cap_{i \in I} F(T_i)}(x_0)$.

Motivated and inspired by the research going on in this direction, the purpose of this paper is to introduce a hybrid projection iterative algorithm and prove strong some convergence theorems for a family of quasi-ϕ-asymptotically nonexpansive mappings in the setting of Banach spaces. The results presented in the paper improve and extend the corresponding results in [2–5].

2. PRELIMINARIES

Let E be a Banach space with a dual E^* and C be a nonempty closed convex subsets of E. We denote by $J : E \to 2^{E^*}$ the normalized duality mapping defined by

$$Jx = \{f \in E^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2\}, \quad x \in E.$$

It is well known that if E is uniformly convex and uniformly smooth, then J and J^{-1} both are uniformly continuous on bounded subsets of E and E^*, respectively.

In the sequel, we always denote by $\phi : E \times E \to R^+$ the Lyapunov functional defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x,\ Jy\rangle + \|y\|^2, \ \ \forall x, y \in E. \tag{2.1}$$

From the definition of ϕ, it is obvious that

$$(\|x\| - \|y\|)^2 \le \phi(x, y) \le (\|x\| + \|y\|)^2, \quad \forall x, y \in E. \tag{2.2}$$

The *generalized projection* $\Pi_C : E \to C$ is defined by

$$\Pi_C(x) = \inf_{y \in C} \phi(y, x), \quad \forall x \in E. \tag{2.3}$$

Lemma 2.1 ([6]). *Let E be a smooth, strict convex and reflexive Banach space and C be a nonempty closed convex subset of E. Then, the following conclusions hold:*

(i) $\phi(x, \Pi_C y) + \phi(\Pi_C y, y) \le \phi(x, y), \quad \forall x \in C, y \in E.$
(ii) *Let $x \in E$ and $z \in C$, then*

$$z = \Pi_C x \iff \langle z - y,\ Jx - Jz\rangle \ge 0, \ \ \forall y \in C. \tag{2.4}$$

Let C be a closed convex subset of E, and T a mapping from C into itself. T is said to be *ϕ-asymptotically nonexpansive*, if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ such that $\phi(T^n x, T^n y) \le k_n \phi(x, y)$ for all $n \ge 1$ and $x, y \in C$. T is said to be *quasi-ϕ-asymptotically nonexpansive*, if $F(T) \ne \emptyset$ and there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ such that $\phi(p, T^n x) \le k_n \phi(p, x)$ for all $n \ge 1$, $x \in C$ and $p \in F(T)$. T is said to be *closed*, if for any $\{x_n\}$ with $x_n \to x$ and $Tx_n \to y$, then we have $Tx = y$.

T is said to be asymptotically regular on C if, for any bounded subset D of C, the following equality holds:

$$\limsup_{n \to \infty} \{\|T^{n+1}x - T^n x\| : x \in D\} = 0.$$

The following lemmas will play an important role in the proof of the main results in this paper.

Lemma 2.2 ([6]). *Let E be a uniformly convex and smooth Banach space and $\{x_n\}$, $\{y_n\}$ be sequences of E. If $\phi(x_n, y_n) \to 0$ (as $n \to \infty$) and either $\{x_n\}$ or $\{y_n\}$ is bounded, then $x_n - y_n \to 0$(as $n \to \infty$).*

Lemma 2.3 ([5]). *Let E be a uniformly convex and smooth Banach space, C be a closed convex subset of E, and T be a closed and quasi-ϕ-asymptotically nonexpansive mapping from C into itself. Then $F(T)$ is a closed convex subset of C.*

Lemma 2.4 ([7]). *Let E be a uniformly convex Banach space, $r > 0$ be a positive number, and $B_r(0)$ be a closed ball of E. For any given points $\{x_1, x_2, \ldots, x_n, \ldots\} \subset B_r(0)$ and for any given positive numbers $\{\lambda_1, \lambda_2, \ldots, \lambda_n, \ldots\}$ with $\Sigma_{n=1}^{\infty} \lambda_n = 1$, there exists a continuous, strictly increasing and convex function $g : [0, 2r) \to [0, \infty)$ with $g(0) = 0$ such that for any $i, j \in \{1, 2, \ldots, \}, i < j$,*

$$\Big\| \sum_{n=1}^{\infty} \lambda_n x_n \Big\|^2 \leq \sum_{n=1}^{\infty} \lambda_n \|x_n\|^2 - \lambda_i \lambda_j g(\|x_i - x_j\|). \tag{2.5}$$

3. MAIN RESULTS

Theorem 3.1. *Let C be a nonempty bounded closed convex subset of a uniformly convex and uniformly smooth Banach space E. For each $i = 1, 2, \ldots$, let $T_i : C \to C$ be a closed and quasi-ϕ-asymptotically nonexpansive mapping with a sequence $\{k_{n,i}\} \subset [1, \infty)$ such that $k_n := \sup_{i \geq 1} k_{n,i} \to 1$ $(n \to \infty)$ and $F = \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$. Suppose further that for each $i = 1, 2, \ldots$, T_i is asymptotically regular on C. Let $\{x_n\}$ be the sequence in C defined by:*

$$\begin{cases} x_0 \in C, \quad C_0 = C, \\ y_n = J^{-1}(\alpha_{n0} J x_n + \sum_{i=1}^{\infty} \alpha_{ni} J T_i^n x_n), \\ C_{n+1} = \{z \in C_n : \phi(z, y_n) \leq \phi(z, x_n) + \xi_n\}, \\ x_{n+1} = \Pi_{C_{n+1}}(x_0), \end{cases} \tag{3.1}$$

where $J : E \to E^*$ is the normalized duality mapping, $M = \sup_{z \in F,\, n \geq 1} \phi(z, x_n)$, $\xi_n = \sum_{i=1}^{\infty} \alpha_{ni}(k_n - 1)M$, and $\{\alpha_{ni}\}$ is the sequence in $[0, 1]$ satisfying the following conditions:

(a) $\sum_{i=0}^{\infty} \alpha_{ni} = 1, \quad \forall n \geq 0$;
(b) $\liminf_{n \to \infty} \alpha_{n0}\, \alpha_{ni} > 0, \quad i = 1, 2, \ldots$.

Then $\{x_n\}$ converges strongly to $\Pi_F x_0$, where Π_F is the generalized projection of E onto F.

Proof. (I) Because $\phi(z, y_n) \leq \phi(z, x_n) + \xi_n$ is equivalent to $2\langle z,\ Jx_n - Jy_n \rangle \leq \|x_n\|^2 - \|y_n\|^2 + \xi_n$. This implies that C_n is a closed and convex subset of C for all $n \geq 0$.

(II) Next, we prove that $F := \bigcap_{i=1}^{\infty} F(T_i) \subset C_n, \ \forall n \geq 0.$

Indeed, it is obvious that $F \subset C_0$. Suppose that $F \subset C_n$ for some $n \in \mathbb{N}$. Noting that $\|\cdot\|^2$ is convex and using (2.1), for any $z \in F \subset C_n$ and for any $\forall m, j \in \{0, 1, 2, \ldots\}$, $m < j$, we have that

$$\begin{aligned}
\phi(z, y_n) &= \phi(z, J^{-1}(\alpha_{n0} J x_n + \sum_{i=1}^{\infty} \alpha_{ni} J T_i^n x_n)) = \\
&= \|z\|^2 - 2\langle z, \ \sum_{i=0}^{\infty} \alpha_{ni} J T_i^n x_n \rangle + \| \sum_{i=0}^{\infty} \alpha_{ni} J T_i^n x_n \|^2 \ (where\ T_0 = I) \leq \\
&\leq \|z\|^2 - \sum_{i=0}^{\infty} \alpha_{ni} 2\langle z, \ J T_i^n x_n \rangle + \sum_{i=0}^{\infty} \alpha_{ni} \|T_i^n x_n\|^2 - \\
&\quad - \alpha_{nm}\alpha_{nj} g(\|J T_m^n x_n - J T_j^n x_n\|) = \\
&= \sum_{i=0}^{\infty} \alpha_{ni} (\|z\|^2 - 2\langle z, \ J T_i^n x_n \rangle + \ \|T_i^n x_n\|^2) - \\
&\quad - \alpha_{nm}\alpha_{nj} g(\|J T_m^n x_n - J T_j^n x_n\|) = \\
&= \alpha_{n0} \phi(z, x_n) + \sum_{i=1}^{\infty} \alpha_{ni} \phi(z, T_i^n x_n) - \alpha_{nm}\alpha_{nj} g(\|J T_m^n x_n - J T_j^n x_n\|) \leq \\
&\leq \alpha_{n0} \phi(z, x_n) + \sum_{i=1}^{\infty} \alpha_{ni} ((k_{n,i} - 1) + 1) \phi(z, x_n) - \\
&\quad - \alpha_{nm}\alpha_{nj} g(\|J T_m^n x_n - J T_j^n x_n\|) \leq \\
&\leq \phi(z, x_n) + \xi_n - \alpha_{nm}\alpha_{nj} g(\|J T_m^n x_n - J T_j^n x_n\|) \leq \phi(z, x_n) + \xi_n.
\end{aligned} \tag{3.2}$$

This implies that $z \in C_n$. Thereby, $F \subset C_n$, $\forall n \geq 0$.

(III) Now, we prove that $\{x_n\}$ is a Cauchy sequence.

Indeed, since $x_{n+1} = \Pi_{C_{n+1}} x_0$ and $x_n = \Pi_{C_n} x_0$, $x_{n+1} \in C_{n+1} \subset C_n$, from the definition of Π_{C_n} we have

$$\phi(x_n, x_0) \leq \phi(x_{n+1}, x_0), \quad \forall n \geq 0. \tag{3.3}$$

Therefore $\{\phi(x_n, x_0)\}$ is nondecreasing. By the assumption that C is bounded, hence from (2.2) we know that $\{\phi(x_n, x_0)\}$ is bounded. This together with (3.3) ensures that the limit $\{\phi(x_n, x_0)\}$ exists. Write

$$\lim_{n\to\infty} \phi(x_n, x_0) = d. \tag{3.4}$$

From Lemma 2.1, we have, for any positive integer $m \geq n$, that

$$\begin{aligned}
\phi(x_m, x_n) = \phi(x_m, \Pi_{C_n} x_0) \leq \phi(x_m, x_0) - \phi(\Pi_{C_n} x_0, x_0) = \\
= \phi(x_m, x_0) - \phi(x_n, x_0)
\end{aligned} \tag{3.5}$$

This implies that

$$\lim_{n,m\to\infty} \phi(x_m, x_n) = 0. \tag{3.6}$$

By Lemma 2.2, we know that $x_m - x_n \to 0$ $(n, m \to \infty)$, hence, $\{x_n\}$ is a Cauchy sequence. Without loss of generality, we can assume that $x_n \to p \in C$ $(n \to \infty)$.

(IV) Now, we prove $\|x_n - T_i^n x_n\| \to 0$ for each $i = 1, 2, \ldots$.

In fact, taking $m = n + 1$ in (3.5) we have that

$$\phi(x_{n+1}, x_n) \le \phi(x_{n+1}, x_0) - \phi(x_n, x_0) \to 0 \ (n \to \infty), \tag{3.7}$$

and hence $x_{n+1} - x_n \to 0$ as $n \to \infty$ by Lemma 2.2. Since $x_{n+1} = \Pi_{C_{n+1}} x_0 \in C_{n+1}$, and by the assumption that $\xi_n \to 0$ (as $n \to \infty$), hence from the definition of C_{n+1}, we have

$$\phi(x_{n+1}, y_n) \le \phi(x_{n+1}, x_n) + \xi_n \to 0 \ (n \to \infty), \tag{3.8}$$

and so $x_{n+1} - y_n \to 0$ $(n \to \infty)$ by Lemma 2.2 Thus we have

$$\|y_n - x_n\| \le \|y_n - x_{n+1}\| + \|x_{n+1} - x_n\| \to 0 \ (n \to \infty). \tag{3.9}$$

Since J is uniformly continuous on any bounded sets of E, we conclude that

$$\lim_{n \to \infty} \|Jx_n - Jy_n\| = 0. \tag{3.10}$$

On the other hand, taking $m = 0$ and $j = 1, 2, \ldots$ in (3.2), for any $z \in F$, we have

$$\phi(z, y_n) \le \phi(z, x_n) + \xi_n - \alpha_{n0}\alpha_{nj} g(\|Jx_n - JT_j^n x_n\|),$$

i.e.,

$$\alpha_{n0}\alpha_{nj} g(\|Jx_n - JT_j^n x_n\|) \le \phi(z, x_n) - \phi(z, y_n) + \xi_n. \tag{3.11}$$

Since

$$\begin{aligned} \phi(z, x_n) - \phi(z, y_n) + \xi_n &= \|x_n\|^2 - \|y_n\|^2 - 2\langle z, Jx_n - Jy_n\rangle + \xi_n \le \\ &\le \|x_n\|^2 - \|y_n\|^2 + 2\|z\|\|Jx_n - Jy_n\| + \xi_n \le \\ &\le \|x_n - y_n\|(\|x_n + y_n\|) + 2\|z\|\|Jx_n - Jy_n\| + \xi_n \end{aligned} \tag{3.12}$$

from (3.9) and (3.10), it follows that $\phi(z, x_n) - \phi(z, y_n) + \xi_n \to 0$ $(n \to \infty)$. Hence, from (3.11) and condition (b) in Theorem 3.1, we have that

$$g(\|Jx_n - JT_j^n x_n\|) \to 0 \ (n \to \infty), \forall j = 1, 2, \ldots \tag{3.13}$$

Since g is continuous and strictly increasing with $g(0) = 0$, it follows from (3.13) that

$$\|Jx_n - JT_j^n x_n\| \to 0 \ (as\ n \to \infty) \quad \text{and for each} \quad j = 1, 2, \ldots$$

Again by the assumption that E is uniformly convex and so E^* is uniformly smooth, hence J^{-1} is uniformly continuous on any bounded subset of E^*. Therefore we have

$$\|x_n - T_j^n x_n\| \to 0 \ (n \to \infty), \quad \text{for each} \quad j = 1, 2, \ldots \tag{3.14}$$

(V) Now, we prove $p \in F$.

From $x_n \to p$ $(n \to \infty)$ and (3.14), we have

$$T_j^n x_n \to p \ (n \to \infty) \quad \text{for each} \quad j = 1, 2, \dots \tag{3.15}$$

Noting that

$$\|T_i^{n+1} x_n - p\| \le \|T_i^{n+1} x_n - T_i^n x_n\| + \|T_i^n x_n - p\|, \tag{3.16}$$

using (3.15) and the asymptotic regularity of T_i, from (3.16) we have

$$T_i^{n+1} x_n \to p \ (n \to \infty), \quad \text{i.e.,} \quad T_i T_i^n x_n \to p \ (n \to \infty). \tag{3.17}$$

By virtue of the closeness of T_i, it follows from (3.15) and (3.17) that p is a fixed point of T_i, $\forall i \ge 1$, i.e., $p \in F$.

(VI) Now, we prove $x_n \to p = \Pi_F x_0$ $(n \to \infty)$.

Let $w = \Pi_F x_0$. From $w \in F \subset C_{n+1}$ and $x_{n+1} = \Pi_{C_{n+1}} x_0$, we have $\phi(x_{n+1}, x_0) \le \phi(w, x_0)$, $\forall n \ge 0$. This implies that

$$\phi(p, x_0) = \lim_{n \to \infty} \phi(x_n, x_0) \le \phi(w, x_0). \tag{3.18}$$

By the definition of $\Pi_F x_0$ and (3.18), we have $p = w$. Therefore, $x_n \to p = \Pi_F x_0$ $(n \to \infty)$.

This completes the proof of theorem 3.1. □

Remark 3.2. *The asymptotic regularity assumption on T_i in Theorem 3.1 can be weakened to the assumption that $T_i^{n+1} x_n - T_i^n x_n \to 0$ as $n \to \infty$. The assumption that $T_i^{n+1} x_n - T_i^n x_n \to 0$ as $n \to \infty$ can be replaced by the uniform Lipschitz continuous of T_i.*

Therefore, we have the following convergence result.

Corollary 3.3. *Let C be a nonempty bounded closed convex subset of a uniformly convex and uniformly smooth Banach space E, and $\{T_i\}_{i=1}^{\infty} : C \to C$ be a family of closed and uniformly Lipschitz continuous and quasi-ϕ-asymptotically nonexpansive mappings with sequence $\{k_{n,i}\}_1^{\infty} \subset [1, \infty)$ such that $k_n := \sup_{i \ge 1} k_{n,i} \to 1$ $(n \to \infty)$ and $F = \bigcap_{i=1}^{\infty} F(T_i) \ne \emptyset$. Let $\{x_n\}$ and $\{\alpha_n\}$ be the same sequences as given in Theorem 3.1 Then $\{x_n\} \to$ converges strongly to $\Pi_F x_0$.*

Because each quasi-ϕ nonexpansive mapping is a quasi-ϕ-asymptotically nonexpansive mapping with sequence $\{k_{n,i} = 1\}$, therefore we have the following

Corollary 3.4. *Let C be a nonempty bounded closed convex subset of a uniformly convex and uniformly smooth Banach space E and $\{T_i\}_{i=1}^{\infty} : C \to C$ be a family of closed and quasi-ϕ nonexpansive mappings such that $F = \bigcap_{i=1}^{\infty} F(T_i) \ne \emptyset$. Let $\{x_n\}$ be the sequence defined by:*

$$\begin{cases} x_0 \in C, \quad C_0 = C, \\ y_n = J^{-1}(\alpha_{n0} J x_n + \sum_{i=1}^{\infty} \alpha_{ni} J T_i x_n), \\ C_{n+1} = \{z \in C_n : \phi(z, y_n) \le \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}}(x_0), \end{cases} \tag{3.19}$$

where $\{\alpha_{ni}\} \subset [0,1]$ *is the sequence satisfying conditions* (a), (b) *in Theorem 3.1. Then* $\{x_n\}$ *converges strongly to* $\Pi_F x_0$.

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J.F. Tang

Yibin University
Department of Mathematics
Yibin, Sichuan 644007, China

S.S. Chang
changss@yahoo.cn

Yibin University
Department of Mathematics
Yibin, Sichuan 644007, China

M. Liu

Yibin University
Department of Mathematics
Yibin, Sichuan 644007, China

J.A. Liu

Beijing Institute of Petro-Chemical Industry
Department of Mathematics and Physics
Beijing, 102617, China

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