BIFURCATION IN A NONLINEAR STEADY STATE SYSTEM

Gen-Qiang Wang, Sui Sun Cheng

Abstract. The steady state solutions of a nonlinear digital cellular neural network with ω neural units and a nonnegative variable parameter λ are sought. We show that $\lambda = 1$ is a critical value such that the qualitative behavior of our network changes. More specifically, when ω is odd, then for $\lambda \in [0, 1)$, there is one positive and one negative steady state, and for $\lambda \in [1, \infty)$, steady states cannot exist; while when ω is even, then for $\lambda \in [0, 1)$, there is one positive and one negative steady state, and for $\lambda = 1$, there are no nontrivial steady states, and for $\lambda \in (1, \infty)$, there are two fully oscillatory steady states. Furthermore, the number of existing nontrivial solutions cannot be improved. It is hoped that our results are of interest to digital neural network designers.

Keywords: bifurcation, cellular neural network, steady state, Krasnoselsky fixed point theorem.

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1. INTRODUCTION

Recurrent cellular neural networks with "circular structure" may yield steady periodic distributions or patterns and hence they are of interests to (digital) neural network designers. For motivation, let us consider a simple prototype model. Let N be the set of nonnegative integers, Z the set of integers, and ω an integer greater than or equal to 2. Let ω neural units $u_1, u_2, \ldots, u_\omega$ be placed on the vertices of a regular polygon in a clockwise manner. The state values of these units are respectively $u_1^{(t)}, \ldots, u_\omega^{(t)}$, where we will take $t \in N$ since we are considering digital devices. The state values may be updated in many possible ways (see e.g. [1–6]). Here we assume that the neural unit on the i-th vertex is forced to change its value by its two "preceding" neighbors in the following manner:

$$u_{i+1}^{(t+1)} = \lambda u_{i-1}^{(t)} + \left(u_i^{(t)}\right)^3,$$

where λ is a nonnegative number. The above equation is valid for $i = 2, 3, \ldots, \omega - 1$ and for $\omega \geq 3$. But the "circular" nature of our network allows us to write

$$\begin{aligned} u_1^{(t+1)} &= \lambda u_{\omega-1}^{(t)} + \left(u_\omega^{(t)}\right)^3, \\ u_2^{(t+1)} &= \lambda u_\omega^{(t)} + \left(u_1^{(t)}\right)^3, \\ \ldots &= \ldots \\ u_\omega^{(t+1)} &= \lambda u_{\omega-2}^{(t)} + \left(u_{\omega-1}^{(t)}\right)^3 \end{aligned}$$

for any $\omega \geq 2$.

To find steady state distributions of our model, that is, time independent solutions of the form $u_i^{(t)} = u_i$, we need to solve the steady state system

$$\begin{aligned} u_1 &= \lambda u_{\omega-1} + (u_\omega)^3, \\ u_2 &= \lambda u_\omega + (u_1)^3, \\ \ldots &= \ldots \\ u_\omega &= \lambda u_{\omega-2} + (u_{\omega-1})^3. \end{aligned} \tag{1.1}$$

Clearly, the existence of such solutions depends on the parameter λ. However, the parity of our model (that is, the parity of the positive integer ω) also plays a crucial role. To see how the partiy affects the existence, let $\lambda = 2$. Then $(u_1, \ldots, u_\omega)^\dagger = (+1, -1, +1, -1, \ldots)^\dagger$ is a solution of our system when ω is even, but fails to be one when ω is odd. For this reason, we will need to consider two major cases (i) ω is even, and (ii) ω is odd. We will show, among other things, that:

(i) ω is even: for $\lambda \in [0, 1)$, a nontrivial solution of (1.1) must either be "positive" or "negative" and our system (1.1) has at least one positive and one negative solution; for $\lambda = 1$, (1.1) does not have any nontrivial solutions; for $\lambda \in (1, +\infty)$, nontrivial solutions of (1.1) must be "fully oscillatory", and (1.1) has at least two such solutions;

(ii) ω is odd: for $\lambda \in [0, 1)$, a nontrivial solution of (1.1) must either be "positive" or "negative" and our system (1.1) has at least one positive and one negative solution; for $\lambda \in [1, +\infty)$, (1.1) does not have any nontrivial solutions.

We will rely on the well known Krasnoselski existence theorem to show our bifurcation results. Actually, this theorem can help us extend our investigations to more general systems of the form

$$\begin{aligned} u_1 &= \lambda u_{\omega-1} + f_\omega(u_\omega), \\ u_2 &= \lambda u_\omega + f_1(u_1), \\ \ldots &= \ldots \\ u_\omega &= \lambda u_{\omega-2} + f_{\omega-1}(u_{\omega-1}), \end{aligned} \tag{1.2}$$

where each f_i has features similar to the cubic function. Indeed, in this paper, we will assume that each f_i satisfies $x f_i(x) > 0$ for $x \neq 0$, and that

$$\lim_{x \to 0} \frac{f_k(x)}{x} = 0, \ \lim_{x \to \pm\infty} \frac{f_k(x)}{x} = +\infty, \ k = 1, 2, \ldots, \omega, \tag{1.3}$$

and we will show that the two assertions above are still true if (1.1) is replaced by (1.2).

To set-up our investigation, we first state the Krasnoselski fixed point theorem in the following form:

Theorem 1.1. *Suppose K is a cone in the Banach space X, Ω_1 and Ω_2 are two bounded open sets in X such that $\theta \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$, and $\Phi \colon K \to K$ is a completely continuous operator. Further, suppose that any one of the following conditions is satisfied:* (i) *for each $u \in K \cap \partial\Omega_1$, $\|\Phi u\| \leq \|u\|$, and for each $u \in K \cap \partial\Omega_2$, $\|\Phi u\| \geq \|u\|$*; (ii) *for each $u \in K \cap \partial\Omega_1$, $\|\Phi u\| \geq \|u\|$, and for each $u \in K \cap \partial\Omega_2$, $\|\Phi u\| \leq \|u\|$, where $\partial\Omega_1$ and $\partial\Omega_2$ represent the boundary of Ω_1 and Ω_2 respectively. Then Φ has a fixed point $u^0 \in K \cap \left(\overline{\Omega}_2 \backslash \Omega_1\right)$.*

For any positive integer ω, we will let R^ω denote the set of all real column vectors endowed with the usual linear operations as well as the norm defined by

$$\|u\| = \sum_{i=1}^{\omega} |u_i| \text{ for } u = (u_1, u_2, \ldots, u_\omega)^\dagger \in R^\omega.$$

R^ω is a well known Banach space. Some of its elements will stand out in our later discussions. In particular, a vector $\varphi = (\varphi_1, \varphi_2, \ldots, \varphi_\omega)^\dagger$ in R^ω is said to be zero-free if $\varphi_m \neq 0$ for $m = 1, \ldots, \omega$, positive if $\varphi_m > 0$ for $m = 1, \ldots, \omega$, negative if $\varphi_m < 0$ for $m = 1, \ldots, \omega$, and fully oscillatory if $\varphi_m \varphi_{m+1} < 0$ for $m = 1, 2, \ldots, \omega - 1$.

A solution of (1.2) is meant to be a vector $\varphi = (\varphi_1, \ldots, \varphi_\omega)^\dagger \in R^\omega$ which renders (1.2) an identiy after substitution into it. Note that if we extend it periodically to an infinite sequence $\{\varphi_i\}_{i\in Z}$ in the following manner

$$\varphi_i = \varphi_{i \bmod \omega}, \ i \in Z, \tag{1.4}$$

then it is a periodic solution of the recurrence relation

$$u_{i+1} = \lambda u_{i-1} + f_i\left(u_i\right), \ i \in Z, \tag{1.5}$$

where

$$f_i = f_{i \bmod \omega}, \ i \in Z. \tag{1.6}$$

Here we recall that a sequence $\{\varphi_i\}_{i\in Z}$ is periodic with period ω if $\varphi_{i+\omega} = \varphi_i$ for $i \in Z$. As for finite vectors, we may define positive, negative, zero free and fully oscillatory sequences in a similar manner.

2. MAIN RESULTS

In order to prove the statements asserted above, we consider various cases depending on λ.

Lemma 2.1. *Suppose $\lambda \geq 1$. Let $\varphi = \{\varphi_m\}_{m\in \mathbf{Z}}$ be a solution of* (1.5).

(i) *If there is an integer* α *such that* $\varphi_\alpha > 0$ *and* $\varphi_{\alpha+1} > 0$, *then* φ *cannot be periodic with period* ω.
(ii) *If there is an integer* α *such that* $\varphi_\alpha < 0$ *and* $\varphi_{\alpha+1} < 0$, *then* φ *cannot be periodic with period* ω.
(iii) *If there is an integer* α *such that* $\varphi_\alpha = 0$ *and* $\varphi_{\alpha+1} > 0$, *then* φ *cannot be periodic with period* ω.
(iv) *If there is an integer* α *such that* $\varphi_\alpha = 0$ *and* $\varphi_{\alpha+1} < 0$, *then* φ *cannot be periodic with period* ω.

Proof. Suppose that $\varphi_\alpha > 0$ and $\varphi_{\alpha+1} > 0$. By (1.5), we see that

$$\varphi_{\alpha+2} = f_{\alpha+1}(\varphi_{\alpha+1}) + \lambda\varphi_\alpha > \lambda\varphi_\alpha \geq \varphi_\alpha,$$
$$\varphi_{\alpha+3} = f_{\alpha+2}(\varphi_{\alpha+2}) + \lambda\varphi_{\alpha+1} > \lambda\varphi_{\alpha+1} \geq \varphi_{\alpha+1}.$$

By induction, we may then see that $\{\varphi_{\alpha+2k}\}_{k=0}^{\infty}$ is a strictly increasing sequence. Thus φ is not a periodic sequence. Similarly, we may see that (ii) is true.

If $\varphi_\alpha = 0$ and $\varphi_{\alpha+1} > 0$, then by (1.5), we see that $\varphi_{\alpha+2} = f_{\alpha+1}(\varphi_{\alpha+1}) + \lambda\varphi_\alpha = f_{\alpha+1}(\varphi_{\alpha+1}) > 0$. Thus $\varphi_{\alpha+1} > 0$ and $\varphi_{\alpha+2} > 0$. We may now apply (i) to conclude our proof. (iv) is similarly proved. □

Lemma 2.2. *Suppose* $\lambda \geq 1$. *If* $\varphi = \{\varphi_i\}_{i\in Z}$ *is a nontrivial solution of* (1.5) *which is periodic with period* ω, *then* ω *must be even and* φ *is fully oscillatory.*

Proof. Suppose $\lambda \geq 1$. Note that if $\varphi = \{\varphi_i\}_{i\in Z}$ is a nontrivial solution of (1.5), then since (1.5) is a three term recurrence relation, $\varphi_\alpha\varphi_{\alpha+1} \neq 0$ for any integer α. If $\varphi = \{\varphi_m\}_{m\in\mathbf{Z}}$ is periodic with period ω, then by Lemma 2.1, it is easy to see that φ is fully oscillatory. Without loss of generality, we can assume that $\varphi_{2k} > 0$ and $\varphi_{2k+1} < 0$ for $k \in Z$. If ω is odd, then $\omega + 1$ is even, so that $\varphi_{\omega+1} > 0$. But by the periodicity of φ, we see that $\varphi_{\omega+1} = \varphi_1 < 0$, which is a contradiction. The proof is complete. □

As an immediate consequence of Lemma 2.2, we have the following nonexistence result.

Theorem 2.3. *Suppose* ω *is odd. If* $\lambda \geq 1$, *then* (1.2) *cannot have any nontrivial solutions.*

Theorem 2.4. *Suppose* ω *is even. If* $\lambda > 1$, *then* (1.2) *has two solutions which are fully oscillatory.*

Proof. Let

$$K = \left\{u = (u_1, u_2, \ldots, u_\omega) \in R^\omega : (-1)^k u_k \geq 0, k = 1, 2, \ldots, \omega\right\}. \tag{2.1}$$

Then K is a cone in R^ω. If we define the mapping $\Phi\colon R^\omega \to R^\omega$ by

$$(\Phi u)_k = -\frac{1}{\lambda} f_{k+1}(u_{k+1}) + \frac{1}{\lambda} u_{k+2},\ k = 1, 2, \ldots, \omega, \tag{2.2}$$

where $u_{\omega+1} = u_1$ and $u_{\omega+2} = u_2$. Then Φ is completely continuous, and it is easy to verify from the definitions of f_k and K that $\Phi K \subset K$. Choose a positive number ε such that $\varepsilon < \lambda - 1$. Then it follows from (1.3) that there exists $\rho_1 > 0$ such that

$$|f_k(x)| \le \varepsilon |x|, \ \ |x| \le \rho_1, \ k = 1, 2, \ldots, \omega. \tag{2.3}$$

In view of (2.2) and (2.3), for each $u \in K$ and $\|u\| = \rho_1$, we have

$$|(\Phi u)_k| \le \frac{1}{\lambda}\left(|f_{k+1}(u_{k+1})| + |u_{k+2}|\right) \le \frac{1}{\lambda}\left(\varepsilon |u_{k+1}| + |u_{k+2}|\right), \ k = 1, 2, \ldots, \omega.$$

It follows that

$$\|\Phi u\| = \sum_{k=1}^{\omega} |(\Phi u)_k| \le \frac{1}{\lambda}\left(\varepsilon \sum_{k=1}^{\omega} |u_{k+1}| + \sum_{k=1}^{\omega} |u_{k+2}|\right) = \frac{1}{\lambda}(1+\varepsilon)\|u\| \le \|u\|. \tag{2.4}$$

In other words, for each $u \in K \cap \partial\Omega_1$, $\|\Phi u\| \le \|u\|$, where $\Omega_1 = \{u \in R^\omega : \|u\| \le \rho_1\}$.

Let M be a positive number such that $M > \lambda + 1$. It follows from (1.3) that there exist $\rho_2 > 0$ and $\rho_2 > \rho_1$ such that

$$|f_k(x)| \ge M |x|, \ k = 1, 2, \ldots, \omega. \tag{2.5}$$

Using (2.2) and (2.5) for each $u \in K$ and $\|u\| = \rho_2$, we have

$$|(\Phi u)_k| \ge \frac{1}{\lambda}\left(|f_{k+1}(u_{k+1})| - |u_{k+2}|\right) \ge \frac{1}{\lambda}\left(M |u_{k+1}| - |u_{k+2}|\right).$$

It follows that

$$\begin{aligned}\|\Phi u\| &= \sum_{k=1}^{\omega} |(\Phi u)_k| \ge \frac{1}{\lambda}\left(M \sum_{k=1}^{\omega} |u_{k+1}| - \sum_{k=1}^{\omega} |u_{k+2}|\right) = \\ &= \frac{1}{\lambda}(M-1)\|u\| \ge \|u\|.\end{aligned} \tag{2.6}$$

That is, for each $u \in K \cap \partial\Omega_2$, $\|\Phi u\| \ge \|u\|$, where $\Omega_2 = \{u \in R^\omega : \|u\| \le \rho_2\}$.

It follows from relations (2.4) and (2.6) and Theorem 1.1 with $\overline{\Omega}_1 \subset \Omega_2$ that the mapping Φ has a fixed point $u^* = (u_1^*, \ldots, u_\omega^*)^\dagger \in K \cap \left(\overline{\Omega}_2 \backslash \Omega_1\right)$. Furthermore, by Theorem 2.1, it is fully oscillatory.

Similarly if we set

$$K' = \left\{u = (u_1, u_2, \ldots, u_\omega) \in R^\omega : (-1)^{k+1} u_k \ge 0, k = 1, 2, \ldots, \omega\right\}, \tag{2.7}$$

then we can find another ω-periodic nontrivial solution which is fully oscillatory. The proof is complete. □

Theorem 2.5. *If* $\lambda = 1$, *then* (1.2) *cannot have any nontrivial solutions.*

Proof. In case ω is odd, then we may see from Theorem 2.3 that our result is true. In case ω is even, let φ be a nontrivial solution of (1.2). Then its periodic extension, which we may write as $\varphi = \{\varphi_m\}_{m\in\mathbf{Z}}$, is a nontrivial solution of (1.5) which is periodic with period ω. By Lemma 2.1, φ is zero-free and fully oscillatory. Without loss of generality, we may assume that $\varphi_{2k} > 0$ and $\varphi_{2k+1} < 0$ for $k \in Z$. From (1.5) we know that for any $k \in Z$, $\varphi_{2k+2} - \varphi_{2k} = f_{2k+1}(\varphi_{2k+1}) < 0$. Thus $\{\varphi_{2k}\}_{k\in\mathbf{Z}}$ is a strictly decreasing sequence. So $\varphi_0 > \varphi_\omega$. But by the periodicity of φ we have $\varphi_0 = \varphi_\omega$. A contradiction is obtained. The proof is complete. □

Lemma 2.6. *Suppose* $0 \le \lambda < 1$. *Let* $\varphi = \{\varphi_m\}_{m\in\mathbf{Z}}$ *be a solution of* (1.5).

(i) *If there is an integer* α *such that* $\varphi_\alpha = 0$ *and* $\varphi_{\alpha+1} > 0$, *then* φ *cannot be periodic with period* ω.

(ii) *If there is an integer* α *such that* $\varphi_\alpha = 0$ *and* $\varphi_{\alpha+1} < 0$, *then* φ *cannot be periodic with period* ω.

Proof. If $\varphi_\alpha = 0$ and $\varphi_{\alpha+1} > 0$, by (1.5) we have

$$\varphi_{\alpha+2} = f_{\alpha+1}(\varphi_{\alpha+1}) + \lambda\varphi_\alpha = f_{\alpha+1}(\varphi_{\alpha+1}) > 0,$$
$$\varphi_{\alpha+3} = f_{\alpha+2}(\varphi_{\alpha+2}) + \lambda\varphi_{\alpha+1} > 0.$$

By induction, we may then see that $\varphi_m > 0$ for $m \ge \alpha + 1$. Thus there cannot be a positive integer ω such that $\varphi_{\alpha+\omega} = \varphi_\alpha = 0$. Similarly, we may also show (ii). □

Theorem 2.7. *Suppose* $0 \le \lambda < 1$. *Then any nontrivial solution of* (1.2) *is either positive or negative.*

Proof. Suppose φ is a nontrivial solution of (1.2). We will prove that φ is either positive or negative. First, its periodic extension, which we may also write as $\varphi = \{\varphi_m\}_{m\in\mathbf{Z}}$ is a nontrivial solution of (1.5) which is periodic with period ω. From Lemma 2.6, we know that φ is zero-free, thus we must consider the following three cases only:

(i) there is an integer α such that $\varphi_\alpha > 0$ and $\varphi_{\alpha+1} > 0$,
(ii) there is an integer α such that $\varphi_\alpha < 0$ and $\varphi_{\alpha+1} < 0$,
(iii) there is an integer α such that $\varphi_{\alpha+2k} > 0$ and $\varphi_{\alpha+2k+1} < 0$ for $k \ge 0$.

In case (i), by (1.5) we have

$$\varphi_{\alpha+2} = f_{\alpha+1}(\varphi_{\alpha+1}) + \lambda\varphi_\alpha > 0,$$
$$\varphi_{\alpha+3} = f_{\alpha+2}(\varphi_{\alpha+2}) + \lambda\varphi_{\alpha+1} > 0.$$

By induction, we may then see that $\varphi_m > 0$ for $m \ge \alpha$. Note that φ has period ω. Thus φ is positive.

Similarly, we may also show that (ii) implies φ is negative.

In case (iii), by (1.5) we have

$$\varphi_{\alpha+2} = f_{\alpha+1}(\varphi_{\alpha+1}) + \lambda\varphi_\alpha < \lambda\varphi_\alpha < \varphi_\alpha,$$
$$\varphi_{\alpha+4} = f_{\alpha+3}(\varphi_{\alpha+3}) + \lambda\varphi_{\alpha+2} < \lambda\varphi_{\alpha+2} < \varphi_{\alpha+2}.$$

By induction, $\{\varphi_{\alpha+2k}\}_{k=0}^{\infty}$ is a strictly decreasing sequence. This is contrary to the periodicity of φ. The proof is complete. □

Theorem 2.8. *If* $0 \le \lambda < 1$, *then* (1.2) *has two nontrivial solutions, one is positive and the other is negative.*

Proof. Let

$$K = \{u = (u_1, u_2, \dots, u_\omega) \in R^\omega : u_k \ge 0, k = 1, 2, \dots, \omega\}. \tag{2.8}$$

Then K is a cone in R^ω. If we define the mapping $\Phi\colon R^\omega \to R^\omega$ by

$$(\Phi u)_k = f_{k-1}(u_{k-1}) + \lambda u_{k-2,}\ k = 1, 2, \dots, \omega, \tag{2.9}$$

where $u_{-1} = u_{\omega-1}$ and $u_0 = u_\omega$, then Φ is completely continuous, and it is easy to verify that $\Phi K \subset K$. Choose a positive number ε such that $\varepsilon + \lambda < 1$. Then from (1.3), there exists a $\rho_1 > 0$ such that

$$|f_k(x)| \le \varepsilon |x|,\ |x| \le \rho_1;\ k = 1, 2, \dots, \omega. \tag{2.10}$$

In view of (2.9) and (2.10), for each $u \in K$ and $\|u\| = \rho_1$, we have

$$|(\Phi u)_k| \le |f_{k-1}(u_{k-1})| + \lambda |u_{k-2}| \le \varepsilon |u_{k-1}| + \lambda |u_{k-2}|,\ k = 1, 2, \dots, \omega. \tag{2.11}$$

It follows that

$$\|\Phi u\| = \sum_{k=1}^{\omega} |(\Phi u)_k| \le \varepsilon \sum_{k=1}^{\omega} |u_{k-1}| + \lambda \sum_{k=1}^{\omega} |u_{k-2}| = (\varepsilon + \lambda)\|u\| \le \|u\|. \tag{2.12}$$

In other words, for each $u \in K \cap \partial\Omega_1$, $\|\Phi u\| \le \|u\|$, where $\Omega_1 = \{u \in R^\omega : \|u\| \le \rho_1\}$. Let M be a positive number such that $M > \lambda + 1$. It follows from (1.3) that there exist $\rho_2 > 0$ and $\rho_2 > \rho_1$ such that

$$|f_k(x)| \ge M|x|,\ k = 1, 2, \dots, \omega. \tag{2.13}$$

Using (2.9) and (2.13) for each $u \in K$ and $\|u\| = \rho_2$, we have

$$|(\Phi u)_k| \ge |f_{k-1}(u_{k-1})| - \lambda |u_{k-2}| \ge M|u_{k-1}| - \lambda |u_{k-2}|.$$

It follows that

$$\|\Phi u\| = \sum_{k=1}^{\omega} |(\Phi u)_k| \ge M \sum_{k=1}^{\omega} |u_{k-1}| - \lambda \sum_{k=1}^{\omega} |u_{k-2}| = (M - \lambda)\|u\| \ge \|u\|. \tag{2.14}$$

That is, for each $u \in K \cap \partial\Omega_2$, $\|\Phi u\| \ge \|u\|$, where $\Omega_2 = \{u \in R^\omega : \|u\| \le \rho_2\}$. It follows from relations (2.4) and (2.6) and Theorem 1.1 with $\overline{\Omega}_1 \subset \Omega_2$ that the mapping Φ has a fixed point $u^* = (u_1^*, \dots, u_\omega^*) \in K \cap (\overline{\Omega}_2 \backslash \Omega_1)$. Furthermore, by Lemma 2.1, φ is zero-free, and hence is a positive solution of (1.5).

Similarly, we can find a vector in

$$K' = \{u = (u_1, u_2, \dots, u_\omega) \in R^\omega : -u_k \ge 0, k = 1, 2, \dots, \omega\} \tag{2.15}$$

which is a negative solution of (1.5). The proof is complete. □

3. EXAMPLES AND REMARKS

As an example, let us consider (1.1). Let $f_k(x) = x^3$ for $k = 1, 2, \ldots, \omega$. Then condition (1.3) is satisfied. By Theorems 2.3–2.5 and 2.7–2.8, we have the results mentioned in the Introduction. In addition, for the special case where $\omega = 2$, our system (1.1) reduces to

$$\begin{aligned} u_1 &= \lambda u_1 + u_2^3, \\ u_2 &= \lambda u_2 + u_1^3, \end{aligned} \tag{3.1}$$

and we may show directly that:

(i) for $\lambda \in (1, \infty)$, all solutions of (3.1) can be found and are given by $(0,0)^\dagger, \left(\pm\sqrt{\lambda-1}, \mp\sqrt{\lambda-1}\right)^\dagger$;
(ii) for $\lambda = 1$, the only solution is $(0,0)^\dagger$;
(iii) for $\lambda \in [0,1)$, all solutions of (3.1) can be found and are given by $(0,0)^\dagger, \left(\pm\sqrt{1-\lambda}, \pm\sqrt{1-\lambda}\right)^\dagger$.

Indeed, if $\lambda = 1$, then from (3.1), we see that $u_2^3 = u_1^3 = 0$, so that $u_1 = u_2 = 0$. If $\lambda \in [0,1) \cup (1,\infty)$, then from (3.1), we have

$$u_2 = \frac{u_1^3}{1-\lambda},$$
$$(1-\lambda)\, u_1 = \left(\frac{u_1^3}{1-\lambda}\right)^3 = \frac{u_1^9}{(1-\lambda)^3}.$$

Hence

$$u_1\left(u_1^8 - (1-\lambda)^4\right) = 0. \tag{3.2}$$

If $\lambda \in [0,1)$, then (3.2) leads us to

$$u_1 = 0, \quad u_2 = 0;$$

and

$$u_1 = \pm\sqrt{1-\lambda}, \quad u_2 = \pm\sqrt{1-\lambda}.$$

If $\lambda > 1$, then (3.2) leads us to

$$u_1 = 0, \quad u_2 = 0;$$

and

$$u_1 = \pm\sqrt{\lambda-1}, \quad u_2 = \mp\sqrt{1-\lambda}.$$

We remark that our previous example shows that when ω is even, our main assertion about (1.2) is "sharp" in the sense that the number of its nontrivial solutions of (1.2) cannot be improved.

For the case where $\omega = 3$, our system reduces to

$$\begin{aligned} u_1 &= \lambda u_2 + u_3^3, \\ u_2 &= \lambda u_3 + u_1^3, \\ u_3 &= \lambda u_1 + u_2^3, \end{aligned} \tag{3.3}$$

and we may show directly that:

(i) for $\lambda \geq 1$, the only solution of (3.3) is $(0,0,0)^{\dagger}$;
(ii) for $\lambda \in [0,1)$, all solutions of (3.3) can be found and are given by $(0,0,0)^{\dagger}$ and

$$\left(\pm\sqrt{1-\lambda}, \pm\sqrt{1-\lambda}, \pm\sqrt{1-\lambda}\right)^{\dagger}.$$

Indeed, suppose $\lambda = 1$. Then from (3.3), we see that

$$u_1^3 + u_2^3 + u_3^3 = 0. \tag{3.4}$$

If one of u_1, u_2, u_3, say u_1, is 0, then from the second equation in (3.3), we see that $u_2 = u_3$, so that substitution of $u_1 = 0$ and $u_2 = u_3$ into (3.4) yields $2u_2^3 = 2u_3^3 = 0$. Hence $u_1 = u_2 = u_3 = 0$. If none of u_1, u_2 or u_3 is 0, then by (3.4) we may assume without loss of generality that at least two components have distinct signs, say, $u_1 > 0$ and $u_2 < 0$. Then from the first equation of (3.3), we see that $u_3^3 > 0$ and hence $u_3 > 0$. But then from the second equation of (3.3), we see that $u_1^3 < 0$, which is a contradiction.

Suppose $\lambda > 1$. If one of u_1, u_2, u_3, say u_1, is 0, then from the first and the second equations of (3.3), we have

$$\begin{aligned} u_2 &= \lambda u_3, \\ u_3\left(\lambda^2 + u_3^2\right) &= 0, \end{aligned}$$

so that $u_3 = 0 = u_2$ and $u_1 = \lambda u_2 + u_3^3 = 0$. If $u_1, u_2, u_3 > 0$, then from (3.3),

$$\begin{aligned} u_1 &= \lambda u_2 + u_3^3 > \lambda u_2 > u_2, \\ u_2 &= \lambda u_3 + u_1^3 > \lambda u_3 > u_3, \\ u_3 &= \lambda u_1 + u_2^3 > \lambda u_1 > u_1, \end{aligned}$$

so that

$$u_1 > u_2 > u_3 > u_1,$$

which is a contradiction. If $u_1, u_2, u_3 < 0$, then as in the previous case, we may show that $u_1 < u_2 < u_3 < u_1$, which is a contradiction. If at least two components have distinct signs, say, $u_1 > 0$ and $u_2 < 0$, then from the first equation of (3.3), we see that $u_3 > 0$. Then from the second equation, we see further that $u_1 < 0$, which is a contradiction.

Suppose $\lambda = 0$. Then from (3.3),

$$u_1 = u_3^3, \quad u_2 = u_1^3, \quad u_3 = u_2^3,$$

so that

$$u_1\left(u_1^{26} - 1\right) = 0.$$

Hence

$$u_1 = 0, \quad u_3 = u_2 = 0;$$

or

$$u_1 = \pm 1, \quad u_3 = \pm 1, \quad u_2 = \pm 1.$$

Finally, suppose $\lambda \in (0,1)$. If one of u_1, u_2, u_3, say u_1 is 0, then from the first and the second equations of (3.3), we have

$$u_2 = \lambda u_3,$$
$$u_3\left(\lambda^2 + u_3^2\right) = 0,$$

so that $u_3 = 0 = u_2$ and $u_1 = \lambda u_2 + u_3^3 = 0$. If two components of $(u_1, u_2, u_3)^\dagger$ are positive, say, $u_1, u_2 > 0$, then from the third equation of (3.3), $u_3 > 0$. Furthermore, we may assert that $u_1, u_2, u_3 \in (0,1)$. Otherwise, assume without loss of generality that $u_3 \geq 1$. Then from (3.3), we have

$$\begin{aligned} u_1 &= \lambda u_2 + u_3^3 \geq u_3^3 \geq u_3 \geq 1, \\ u_2 &= \lambda u_3 + u_1^3 > u_1^3 \geq u_1 \geq 1, \\ u_3 &= \lambda u_1 + u_2^3 > u_2^3 \geq u_2, \end{aligned}$$

so that $u_1 \geq u_3 > u_2 > u_1$, which is a contradiction. Now that $u_1, u_2, u_3 \in (0,1)$, we assert further that $u_1 = u_2 = u_3$. Indeed, assume without loss of generality that $u_1 = \max\{u_1, u_2, u_3\}, u_2 = \min\{u_1, u_2, u_3\}$ and $u_1 > u_2$. Then from the first equation of (3.3), we have

$$u_1 = \lambda u_2 + u_3^3 < \lambda u_1 + u_1^3,$$

so that

$$u_1 > \sqrt{1-\lambda}. \tag{3.5}$$

By the second equation of (3.3), we see that

$$u_2 = \lambda u_3 + u_1^3 > \lambda u_2 + u_2^3, \tag{3.6}$$

so that

$$u_2 < \sqrt{1-\lambda}. \tag{3.7}$$

On the other hand, by the second equation of (3.3) and (3.5) we see that

$$u_2 = \lambda u_3 + u_1^3 > \lambda u_2 + (1-\lambda)\sqrt{1-\lambda},$$

thus

$$u_2 > \sqrt{1-\lambda}, \tag{3.8}$$

which is a contradiction. Now that $u_1 = u_2 = u_3 \in (0,1)$, we may easily see from (3.3) that

$$u_1 = u_2 = u_3 = \sqrt{1-\lambda}.$$

Similarly, if two components of $(u_1, u_2, u_3)^\dagger$ are negative, we may show in a similar way that

$$u_1 = u_2 = u_3 = -\sqrt{1-\lambda}.$$

As a consequence of the above example, we see that our main assertions regarding the number of nontrivial solutions of (1.2) are also sharp when ω is odd.

As our final remark, if we change the conditions on f_k in (1.3) to

$$\lim_{x\to 0}\frac{f_k(x)}{x}=+\infty,\quad \lim_{x\to\pm\infty}\frac{f_k(x)}{x}=0,\ k=1,2,\dots,\omega, \tag{3.9}$$

then by duality considerations, we may easily obtain the following (sharp) assertions:

(i) ω is even: for $\lambda\in[0,1)$, a nontrivial solution of (1.2) must either be positive or negative and our system (1.2) has at least one positive and one negative solution; for $\lambda=1$, (1.2) does not have any nontrivial solutions; for $\lambda\in(1,+\infty)$, nontrivial solutions of (1.2) must be fully oscillatory, and (1.2) has at least two such solutions;

(ii) ω is odd: for $\lambda\in[0,1)$, a nontrivial solution of (1.2) must either be positive or negative and our system (1.2) has at least one positive and one negative solution; for $\lambda\in[1,+\infty)$, (1.1) does not have any nontrivial solutions.

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Gen-Qiang Wang
w7633@hotmail.com

Guangdone Polytechnic Normal University
Department of Computer Science
Guangzhou, Guangdone 510665, P. R. China

Sui Sun Cheng (corresponding author)
sscheng@math.nthu.edu.tw

Tsing Hua University
Department of Mathematics
Hsinchu, Taiwan 30043, R. O. China

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