AN ALGORITHM FOR FINDING A COMMON SOLUTION FOR A SYSTEM OF MIXED EQUILIBRIUM PROBLEM, QUASI-VARIATIONAL INCLUSION PROBLEM AND FIXED POINT PROBLEM OF NONEXPANSIVE SEMIGROUP

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Abstract. In this paper, we introduce a hybrid iterative scheme for finding a common element of the set of solutions for a system of mixed equilibrium problems, the set of common fixed points for a nonexpansive semigroup and the set of solutions of the quasi-variational inclusion problem with multi-valued maximal monotone mappings and inverse-strongly monotone mappings in a Hilbert space. Under suitable conditions, some strong convergence theorems are proved. Our results extend some recent results in the literature.

Keywords: nonexpansive semigroup, mixed equilibrium problem, viscosity approximation method, quasi-variational inclusion problem, multi-valued maximal monotone mappings, α-inverse-strongly monotone mapping.

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1. INTRODUCTION

Throughout this paper we assume that H is a real Hilbert space and C is a nonempty closed convex subset of H.

In the sequel, we denote the set of fixed points of a mapping S by $F(S)$.

A bounded linear operator $A : H \to H$ is said to be *strongly positive*, if there exists a constant $\bar{\gamma}$ such that

$$\langle Ax, x\rangle \geq \bar{\gamma}\|x\|^2, \quad \forall\, x \in H. \tag{1.1}$$

Let $B : H \to H$ be a single-valued nonlinear mapping and $M : H \to 2^H$ be a multi-valued mapping. The "so-called" *quasi-variational inclusion problem* (see, Chang [2, 3]) is to find an $u \in H$ such that

$$\theta \in B(u) + M(u). \tag{1.2}$$

A number of problems arising in structural analysis, mechanics and economics can be studied in a framework of this kind of variational inclusions (see, for example [5]).

The set of solutions of quasi-variational inclusion (1.2) is denoted by **VI(H,B,M)**.

Special Case

If $M = \partial\delta_C$, where $\partial\delta_C$ is the subdifferential of δ_C, C is a nonempty closed convex subset of H and $\delta_C : H \to [0, \infty)$ is the indicator function of C, i.e.,

$$\delta_C = \begin{cases} 0, & x \in C, \\ +\infty, & x \notin C, \end{cases}$$

then the quasi-variational inclusion problem (1.2) is equivalent to find $u \in C$ such that

$$\langle B(u), v - u\rangle \geq 0, \ \ \forall v \in C. \tag{1.3}$$

This problem is called the *Hartman-Stampacchia variational inequality problem* (see, for example [7]). The set of solutions of (1.3) is denoted by **VI(C, B)**.

Recall that a mapping $B : H \to H$ is called *α-inverse strongly monotone* (see [13]), if there exists an $\alpha > 0$ such that

$$\langle Bx - By, x - y\rangle \geq \alpha\|Bx - By\|^2, \ \ \forall x, y \in H.$$

A multi-valued mapping $M : H \to 2^H$ is called *monotone*, if for all $x, y \in H, u \in Mx$, and $v \in My$, implies that $\langle u-v, x-y\rangle \geq 0$. A multi-valued mapping $M : H \to 2^H$ is called *maximal monotone*, if it is monotone and if for any $(x, u) \in H \times H$

$$\langle u - v, \ x - y\rangle \geq 0, \ \ \forall\ (y, v) \in Graph(M)$$

(the graph of mapping M) implies that $u \in Mx$.

Proposition 1.1 ([13]). *Let $B : H \to H$ be an α-inverse strongly monotone mapping, then:*

(a) *B is $\frac{1}{\alpha}$-Lipschitz continuous and a monotone mapping;*
(b) *If λ is any constant in $(0, 2\alpha]$, then the mapping $I - \lambda B$ is nonexpansive, where I is the identity mapping on H.*

Let C be a nonempty closed convex subset of H, $\Theta : C \times C \to R$ be an equilibrium bifunction (i.e., $\Theta(x, x) = 0, \ \forall x \in C$) and let $\varphi : C \to R$ be a real-valued function.

Recently, Ceng and Yao [1] introduced the following *mixed equilibrium problem* (MEP), i.e., to find $z \in C$ such that

$$MEP : \Theta(z, y) + \varphi(y) - \varphi(z) \geq 0, \ \forall y \in C. \tag{1.4}$$

The set of solutions of (1.4) is denoted by $MEP(\Theta, \varphi)$, i.e.,

$$MEP(\Theta) = \{z \in C : \Theta(z, y) + \varphi(y) - \varphi(z) \geq 0, \ \forall y \in C\}.$$

In particular, if $\varphi = 0$, this problem reduces to the *equilibrium problem*, i.e., to find $z \in C$ such that

$$EP: \quad \Theta(z, y) \geq 0, \ \forall y \in C.$$

Denote the set of solution of EP by $EP(\Theta)$.

On the other hand, Li *et al.* [6] introduced a two step iterative procedure for the approximation of common fixed points of a nonexpansive semigroup $\{T(s) : 0 \leq s < \infty\}$ on a nonempty closed convex subset C in a Hilbert space.

Very recently, Saeidi [9] introduced a more general iterative algorithm for finding a common element of the set of solutions for a system of equilibrium problems and of the set of common fixed points for a finite family of nonexpansive mappings and a nonexpansive semigroup.

Recall that a family of mappings $\mathscr{T} = \{T(s) : 0 \leq s < \infty\} : C \to C$ is called *a nonexpansive semigroup*, if it satisfies the following conditions:

(a) $T(s + t) = T(s)T(t)$ for all $s, t \geq 0$ and $T(0) = I$;
(b) $\|T(s)x - T(s)y\| \leq \|x - y\|$, $\forall x, y \in C$.
(c) The mapping $T(\cdot)x$ is continuous, for each $x \in C$.

Motivated and inspired by Ceng and Yao [1], Li *et al.* [6] and Saeidi [9], the purpose of this paper is to introduce a hybrid iterative scheme for finding a common element of the set of solutions for a system of mixed equilibrium problems, the set of common fixed points for a nonexpansive semigroup and the set of solutions of the quasi-variational Inclusion problem with multi-valued maximal monotone mappings and inverse-strongly monotone mappings in Hilbert space. Under suitable conditions, some strong convergence theorems are proved. Our results extends the recent results in Zhang, Lee and Chan [13], Takahashi and Takahashi [11], Chang, Joseph Lee and Chan [4], Ceng and Yao [1], Li *et al.* [6] and Saeidi [9].

2. PRELIMINARIES

In the sequel, we use $x_n \rightharpoonup x$ and $x_n \to x$ to denote the weak convergence and strong convergence of the sequence $\{x_n\}$ in H, respectively.

Definition 2.1. Let $M : H \to 2^H$ be a multi-valued maximal monotone mapping, then the single-valued mapping $J_{M,\lambda} : H \to H$ defined by

$$J_{M,\lambda}(u) = (I + \lambda M)^{-1}(u), \ \ \forall u \in H$$

is called the *resolvent operator associated with* M, where λ is any positive number and I is the identity mapping.

Proposition 2.2 ([13])**.** (a) *The resolvent operator $J_{M,\lambda}$ associated with M is single-valued and nonexpansive for all $\lambda > 0$, i.e.,*

$$\|J_{M,\lambda}(x) - J_{M,\lambda}(y)\| \leq \|x - y\|, \ \ \forall x, y \in H, \ \forall \lambda > 0.$$

(b) *The resolvent operator $J_{M,\lambda}$ is 1-inverse-strongly monotone, i.e.,*

$$\|J_{M,\lambda}(x) - J_{M,\lambda}(y)\|^2 \leq \langle x - y, \ J_{M,\lambda}(x) - J_{M,\lambda}(y)\rangle, \ \ \forall x, y \in H.$$

Definition 2.3. A single-valued mapping $P : H \to H$ is said to be *hemi-continuous*, if for any $x, y \in H$, the mapping $t \mapsto P(x + ty)$ converges weakly to Px (as $t \to 0+$).

It is well-known that every continuous mapping must be hemi-continuous.

Lemma 2.4 ([8])**.** *Let E be a real Banach space, E^* be the dual space of E, $T : E \to 2^{E^*}$ be a maximal monotone mapping and $P : E \to E^*$ be a hemi-continuous bounded monotone mapping with $D(P) = E$ then the mapping $S = T + P : E \to 2^{E^*}$ is a maximal monotone mapping.*

For solving the equilibrium problem for bifunction $\Theta : C \times C \to R$, let us assume that Θ satisfies the following conditions:
(H_1) $\Theta(x, x) = 0$ for all $x \in C$.
(H_2) Θ is monotone, i.e., $\Theta(x, y) + \Theta(y, x) \leq 0$ for all $x, y \in C$.
(H_3) For each $y \in C$, $x \mapsto \Theta(x, y)$ is concave and upper semicontinuous.
(H_4) For each $x \in C$, $y \mapsto \Theta(x, y)$ is convex.

A map $\eta : C \times C \to H$ is called Lipschitz continuous, if there exists a constant $L > 0$ such that

$$\|\eta(x, y)\| \leq L\|x - y\|, \ \ \forall x, y \in C.$$

A differentiable function $K : C \to R$ on a convex set C is called:

(i) η-convex [1] if

$$K(y) - K(x) \geq \langle K'(x), \eta(y, x)\rangle, \ \ \forall x, y \in C,$$

where $K'(x))$ is the *Fréchet* derivative of K at x;

(ii) η-strongly convex[7] if there exists a constant $\mu > 0$ such that

$$K(y) - K(x) - \langle K'(x), \eta(y, x)\rangle \geq \Big(\frac{\mu}{2}\Big)\|x - y\|^2, \ \ \forall x, y \in C.$$

Let $\Theta : C \times C \to R$ be an equilibrium bifunction satisfying the conditions $(H_1) - (H_4)$. Let r be any given positive number. For a given point $x \in C$, consider the following *auxiliary problem for MEP* (for short, $MEP(x, r)$): to find $y \in C$ such that

$$\Theta(y, z) + \varphi(z) - \varphi(y) + \frac{1}{r}\langle K'(y) - K'(x), \eta(z, y)\rangle \geq 0, \ \ \forall z \in C,$$

where $\eta : C \times C \to H$ is a mapping and $K'(x)$ is the *Fréchet* derivative of a functional $K : C \to R$ at x. Let $V_r^{\Theta} : C \to C$ be the mapping such that for each $x \in C$, $V_r^{\Theta}(x)$ is the solution set of $MEP(x, r)$, i.e.,

$$\begin{aligned} V_r^{\Theta}(x) = \{y \in C : \Theta(y, z) + \varphi(z) - \varphi(y) + \\ + \frac{1}{r}\langle K'(y) - K'(x), \eta(z, y)\rangle \geq 0, \ \ \forall z \in C\}, \ \ \forall x \in C. \end{aligned} \tag{2.1}$$

Then the following conclusion holds:.

Proposition 2.5 ([1]). *Let C be a nonempty closed convex subset of H, $\varphi : C \to R$ be a lower semicontinuous and convex functional. Let $\Theta : C \times C \to R$ be an equilibrium bifunction satisfying conditions* (H_1)–(H_4). *Assume that:*

(i) $\eta : C \times C \to H$ *is Lipschitz continuous with constant* $L > 0$ *such that:*
  (a) $\eta(x, y) + \eta(y, x) = 0,\ \forall x, y \in C$,
  (b) $\eta(\cdot, \cdot)$ *is affine in the first variable,*
  (c) *for each fixed* $y \in C, x \mapsto \eta(y, x)$ *is continuous from the weak topology to the weak topology;*

(ii) $K : C \to R$ *is* η*-strongly convex with constant* $\mu > 0$ *and its derivative* K' *is continuous from the weak topology to the strong topology;*

(iii) *for each* $x \in C$*, there exist a bounded subset* $D_x \subseteq C$ *and* $z_x \in C$ *such that for any* $y \in C \setminus D_x$*, the following holds:*

$$\Theta(y, z_x) + \varphi(z_x) - \varphi(y) + \frac{1}{r}\langle K'(y) - K'(x), \eta(z_x, y)\rangle < 0.$$

Then the following holds:

(i) V_r^{Θ} *is single-valued;*
(ii) V_r^{Θ} *is nonexpansive if* K' *is Lipschitz continuous with constant* $\nu > 0$ *such that* $\mu \geq L\nu$;
(iii) $F(V_r^{\Theta}) = MEP(\Theta)$;
(iv) $MEP(\Theta)$ *is closed and convex.*

Lemma 2.6 ([10]). *Let C be a nonempty bounded closed convex subset of H and let $\Im = \{T(s) : 0 \leq s < \infty\}$ be a nonexpansive semigroup on C, then for any $h \geq 0$.*

$$\lim_{t\to\infty} \sup_{x \in C} \|\frac{1}{t}\int_0^t T(s)x ds - T(h)(\frac{1}{t}\int_0^t T(s)x ds)\| = 0.$$

Lemma 2.7 ([6]). *Let C be a nonempty bounded closed convex subset of H and let $\Im = \{T(s) : 0 \leq s < \infty\}$ be a nonexpansive semigroup on C. If $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup z$ and* $\limsup_{s\to\infty} \limsup_{n\to\infty} \|T(s)x_n - x_n\| = 0$, *then* $z \in F(\Im)$.

3. MAIN RESULTS

In order to prove the main result, we first give the following Lemma.

Lemma 3.1 ([13]). (a) *$u \in H$ is a solution of variational inclusion* (1.2) *if and only if $u = J_{M,\lambda}(u - \lambda Bu),\ \forall \lambda > 0$, i.e.,*

$$VI(H, B, M) = F(J_{M,\lambda}(I - \lambda B)),\ \forall \lambda > 0.$$

(b) *If $\lambda \in (0, 2\alpha]$, then $VI(H, B, M)$ is a closed convex subset in H.*

In the sequel, we assume that $H, C, M, A, B, f, \mathscr{T}, \mathscr{F}, \varphi_i, \eta_i, K_i (i = 1, 2, \cdots N)$ satisfy the following conditions:

(1) H is a real Hilbert space, $C \subset H$ is a nonempty closed convex subset;
(2) $A : H \to H$ is a strongly positive linear bounded operator with coefficient $\bar{\gamma} > 0$, $f : H \to H$ is a contraction mapping with a contraction constant h $(0 < h < 1)$ and $0 < \gamma < \frac{\bar{\gamma}}{h}$, $B : C \to H$ is a α-inverse-strongly monotone mapping and $M : H \to 2^H$ is a multi-valued maximal monotone mapping;
(3) $\mathscr{T} = \{T(s) : 0 \le s < \infty\} : C \to C$ is a nonexpansive semigroup;
(4) $\mathscr{F} = \{\Theta_i : i = 1, 2, \cdots, N\} : C \times C \to R$ is a finite family of bifunctions satisfying conditions $(H_1) - (H_4)$ and $\varphi_i : C \to R (i = 1, 2, \cdots, N)$ is a finite family of lowersemi-continuous and convex functionals;
(5) $\eta_i : C \times C \to H$ is a finite family of Lipschitz continuous mappings with constant $L_i > 0 (i = 1, 2, \cdots, N)$ such that:
   (a) $\eta_i(x, y) + \eta_i(y, x) = 0, \ \forall x, y \in C$,
   (b) $\eta_i(\cdot, \cdot)$ is affine in the first variable,
   (c) for each fixed $y \in C, x \mapsto \eta_i(y, x)$ is continuous from the weak topology to the weak topology;
(6) $K_i : C \to R$ is a finite family of η_i-strongly convex with constant $\mu_i > 0$ and its derivative K_i' is not only continuous from the weak topology to the strong topology but also Lipschitz continuous with constant $\nu_i > 0, \ \mu_i \ge L_i \nu_i$.

In the sequel we always denote by $F(\mathscr{T})$ the set of fixed points of the nonexpansive semi-group $\mathscr{T}$, VI(H,B,M) the set of solutions to the variational inequality (1.2) and MEP($\mathscr{F}$) the set of solutions to the following *auxiliary problem for a system of mixed equilibrium problems*:

$$\begin{cases} \Theta_1(y_n^{(1)}, x) + \phi_1(x) - \phi_1(y_n^{(1)}) + \dfrac{1}{r_1}\langle K'(y_n^{(1)}) - K'(x_n), \ \eta_1(x, y_n^{(1)})\rangle \ge 0, \ \ \forall x \in C, \\ \Theta_2(y_n^{(2)}, x) + \phi_2(x) - \phi_2(y_n^{(2)}) + \dfrac{1}{r_2}\langle K'(y_n^{(2)}) - K'(y_n^{(1)}), \ \eta_2(x, y_n^{(2)})\rangle \ge 0, \ \ \forall x \in C, \\ \quad \vdots \\ \Theta_{N-1}(y_n^{(N-1)}, x) + \phi_{N-1}(x) - \phi_{N-1}(y_n^{(N-1)}) + \\ \quad + \dfrac{1}{r_{N-1}}\langle K'(y_n^{(N-1)}) - K'(y_n^{(N-2)}), \ \eta_{N-1}(x, y_n^{(N-1)})\rangle \ge 0, \ \ \forall x \in C, \\ \Theta_N(y_n, x) + \phi_N(x) - \phi_N(y_n) + \\ \quad + \dfrac{1}{r_N}\langle K'(y_n) - K'(y_n^{(N-1)}), \ \eta_N(x, y_n)\rangle \ge 0, \ \ \forall x \in C, \end{cases}$$

where

$$\begin{cases} y_n^{(1)} = V_{r_1}^{\Theta_1} x_n, \\ y_n^{(i)} = V_{r_i}^{\Theta_i} y_n^{(i-1)} = V_{r_i}^{\Theta_i} V_{r_{(i-1)}}^{\Theta_{i-1}} y_n^{(i-2)} = V_{r_i}^{\Theta_i} \cdots V_{r_2}^{\Theta_2} y_n^{(1)} \\ \qquad = V_{r_i}^{\Theta_i} \cdots V_{r_2}^{\Theta_2} V_{r_1}^{\Theta_1} x_n, \ \ i = 2, 3, \cdots, N-1, \\ y_n = V_{r_N}^{\Theta_N} \cdots V_{r_2}^{\Theta_2} V_{r_1}^{\Theta_1} x_n, \end{cases}$$

and $V_{r_i}^{\Theta_i}: C \to C, \;\; i = 1, 2, \cdots, N$ is the mapping defined by (2.1)

In the sequel we denote by $\mathscr{V}^l = V_{r_l}^{\Theta_l} \cdots V_{r_2}^{\Theta_2} V_{r_1}^{\Theta_1}$ for $l \in \{1, 2, \cdots, N\}$ and $\mathscr{V}^0 = I$.

Theorem 3.2. *Let $H, C, A, B, M, f, \mathscr{T}, \mathscr{F}, \varphi_i, \eta_i, K_i (i = 1, 2, \cdots, N)$ be the same as above. Let $\{x_n\}, \{\rho_n\}, \{\xi_n\}$ and $\{y_n\}$ be the explicit iterative sequences generated by $x_1 \in H$ and*

$$\begin{cases} x_{n+1} = \alpha_n \gamma f(\frac{1}{t_n} \int_0^{t_n} T(s) x_n ds) + \beta_n x_n + ((1 - \beta_n) I - \alpha_n A) \frac{1}{t_n} \int_0^{t_n} T(s) \rho_n ds, \\ \rho_n = J_{M,\lambda}(I - \lambda B) \xi_n, \qquad\qquad\qquad\qquad \forall\, n \geq 1, \\ \xi_n = J_{M,\lambda}(I - \lambda B) y_n, \\ y_n = V_{r_N}^{\Theta_N} \cdots V_{r_2}^{\Theta_2} V_{r_1}^{\Theta_1} x_n \end{cases} \tag{3.1}$$

where $r_i (i = 1, 2, \cdots, N)$ be a finite family of positive numbers, $\lambda \in (0, 2\alpha]$, $\{\alpha_n\}, \{\beta_n\} \subset [0, 1]$ and $\{t_n\} \subset (0, \infty)$ is a sequence with $t_n \uparrow \infty$. If $\mathscr{G} := F(\mathscr{T}) \bigcap MEP(\mathscr{F}) \bigcap VI(H, B, M) \neq \emptyset$ and the following conditions are satisfied:

(i) *for each $x \in C$, there exist a bounded subset $D_x \subseteq C$ and $z_x \in C$ such that for any $y \in C \setminus D_x$,*

$$\Theta_i(y, z_x) + \varphi_i(z_x) - \varphi_i(y) + \frac{1}{r_i} \langle K_i'(y) - K_i'(x), \eta_i(z_x, y) \rangle < 0.$$

(ii) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$, *then the sequence $\{x_n\}$ converges strongly to $x^* = P_{\mathscr{G}}(I - A + \gamma f)(x^*)$, provided that $V_{r_i}^{\Theta_i}$ is firmly nonexpansive where $P_{\mathscr{G}}$ is the metric projection of H onto $\mathscr{G}$.*

Proof. We observe that from conditions (ii), we can assume, without loss of generality, that $\alpha_n \leq (1 - \beta_n) \|A\|^{-1}$.

Since A is a linear bounded self-adjoint operator on H, then

$$\|A\| = \sup\{|\langle Au, u \rangle| : u \in H, \|u\| = 1\}.$$

Since

$$\langle ((1 - \beta_n) I - \alpha_n A) u, u \rangle = 1 - \beta_n - \alpha_n \langle Au, u \rangle \geq 1 - \beta_n - \alpha_n \|A\| \geq 0,$$

this implies that $(1 - \beta_n) I - \alpha_n A$ is positive. Hence we have

$$\begin{aligned} \|(1 - \beta_n) I - \alpha_n A\| &= \sup\{|\langle ((1 - \beta_n) I - \alpha_n A) u, u \rangle| : u \in H, \|u\| = 1\} \\ &= \sup\{1 - \beta_n - \alpha_n \langle Au,\ u \rangle : u \in H, \|u\| = 1\} \leq \\ &\leq 1 - \beta_n - \alpha_n \bar{\gamma} < 1. \end{aligned}$$

Let $Q = P_{\mathscr{G}}$. Note that f is a contraction with coefficient $h \in (0,1)$. Then, we have

$$\begin{aligned}\|Q(I-A+\gamma f)(x) - Q(I-A+\gamma f)(y)\| &\le \|(I-A+\gamma f)(x) - (I-A+\gamma f)(y)\| \le \\ &\le \|I-A\|\|x-y\| + \gamma\|f(x)-f(y)\| \le \\ &\le (1-\bar{\gamma})\|x-y\| + \gamma h\|x-y\| = \\ &= (1-(\bar{\gamma}-\gamma h))\|x-y\|,\end{aligned}$$

for all $x, y \in H$. Therefore, $Q(I-A+\gamma f)$ is a contraction of H into itself, which implies that there exists a unique element $x^* \in H$ such that $x^* = Q(I-A+\gamma f)(x^*) = P_{\mathscr{G}}(I-A+\gamma f)(x^*)$.

Next, we divide the proof of Theorem 3.2 into 9 steps:

Step 1. First prove the sequences $\{x_n\}$, $\{\rho_n\}$, $\{\xi_n\}$ and $\{y_n\}$ are bounded.

(a) Pick $p \in \mathscr{G}$, since $y_n = \mathscr{V}^N x_n$ and $p = \mathscr{V}^N p$, we have

$$\|y_n - p\| = \|\mathscr{V}^N x_n - p\| \le \|x_n - p\|. \tag{3.2}$$

(b) Since $p \in VI(H,B,M)$ and $\rho_n = J_{M,\lambda}(I-\lambda B)\xi_n$, we have $p = J_{M,\lambda}(I-\lambda B)p$, and so

$$\begin{aligned}\|\rho_n - p\| &= \|J_{M,\lambda}(I-\lambda B)\xi_n - J_{M,\lambda}(I-\lambda B)p\| \le \\ &\le \|(I-\lambda B)\xi_n - (I-\lambda B)p\| \le \|\xi_n - p\| = \\ &= \|J_{M,\lambda}(I-\lambda B)y_n - J_{M,\lambda}(I-\lambda B)p\| \le \\ &\le \|y_n - p\| \le \|x_n - p\|.\end{aligned} \tag{3.3}$$

Letting $u_n = \frac{1}{t_n}\int_0^{t_n} T(s)x_n ds$, $\quad q_n = \frac{1}{t_n}\int_0^{t_n} T(s)\rho_n ds$, we have

$$\|u_n - p\| = \|\frac{1}{t_n}\int_0^{t_n} T(s)x_n ds - p\| \le \frac{1}{t_n}\int_0^{t_n} \|T(s)x_n - T(s)p\| ds \le \|x_n - p\|. \tag{3.4}$$

Similarly, we have

$$\|q_n - p\| \le \|\rho_n - p\|. \tag{3.5}$$

Form (3.1), (3.2), (3.3), (3.4)and (3.5), we have

$$\begin{aligned}&\|x_{n+1} - p\| = \\ &= \|\alpha_n \gamma f(u_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A)q_n - p\| = \\ &= \|\alpha_n\gamma(f(u_n) - f(p)) + \beta_n(x_n - p) + \\ &\quad + ((1-\beta_n)I - \alpha_n A)(q_n - p) + \alpha_n(\gamma f(p) - Ap)\| \le \\ &\le \alpha_n\gamma h\|u_n - p\| + \beta_n\|x_n - p\| + ((1-\beta_n) - \alpha_n\bar{\gamma})\|q_n - p\| + \alpha_n\|\gamma f(p) - Ap\| \le \\ &\le \alpha_n\gamma h\|x_n - p\| + \beta_n\|x_n - p\| + ((1-\beta_n) - \alpha_n\bar{\gamma})\|x_n - p\| + \alpha_n\|\gamma f(p) - Ap\| \le \\ &\le (1 - \alpha_n(\bar{\gamma} - \gamma h))\|x_n - p\| + \alpha_n\|\gamma f(p) - Ap\| \le \\ &\le max\|x_n - p\|, \frac{1}{\bar{\gamma} - \gamma h}\|\gamma f(p) - Ap\| \\ &\vdots \\ &\le max\|x_1 - p\|, \frac{1}{\bar{\gamma} - \gamma h}\|\gamma f(p) - Ap\|.\end{aligned}$$

This implies that $\{x_n\}$ is a bounded sequence in H. Therefore $\{y_n\}$, $\{\rho_n\}$, $\{\xi_n\}$, $\{\gamma f(u_n)\}$ and $\{q_n\}$ are all bounded.
Step 2. Next we prove that

$$\|x_{n+1} - x_n\| \to 0 \ (n \to \infty). \tag{3.6}$$

In fact, let us define a sequence $\{z_n\}$ by

$$x_{n+1} = (1 - \beta_n)z_n + \beta_n x_n \qquad \forall n \geq 1,$$

then we have

$$\begin{aligned}
&z_{n+1} - z_n = \\
&= \frac{x_{n+2} - \beta_{n+1}x_{n+1}}{1 - \beta_{n+1}} - \frac{x_{n+1} - \beta_n x_n}{1 - \beta_n} = \\
&= \frac{\alpha_{n+1}\gamma f(u_{n+1}) + ((1 - \beta_{n+1})I - \alpha_{n+1}A)q_{n+1}}{1 - \beta_{n+1}} - \\
&\quad - \frac{\alpha_n \gamma f(u_n) + ((1 - \beta_n)I - \alpha_n A)q_n}{1 - \beta_n} = \\
&= \frac{\alpha_{n+1}}{1 - \beta_{n+1}}[\gamma f(u_{n+1}) - Aq_{n+1}] - \frac{\alpha_n}{1 - \beta_n}[\gamma f(u_n) - Aq_n] + q_{n+1} - q_n
\end{aligned}$$

and so

$$\begin{aligned}
&\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\| \leq \\
&\leq \frac{\alpha_{n+1}}{1 - \beta_{n+1}}(\|\gamma f(u_{n+1})\| + \|Aq_{n+1}\|) + \\
&\quad + \frac{\alpha_n}{1 - \beta_n}(\|\gamma f(u_n)\| + \|Aq_n\|) + \|\frac{1}{t_{n+1}}\int_0^{t_{n+1}} T(s)\rho_{n+1}ds - \frac{1}{t_{n+1}}\int_0^{t_{n+1}} T(s)\rho_n ds\| + \\
&\quad + \|\frac{1}{t_{n+1}}\int_0^{t_{n+1}} T(s)\rho_n ds - \frac{1}{t_n}\int_0^{t_n} T(s)\rho_n ds\| - \|x_{n+1} - x_n\| \leq \\
&\leq \frac{\alpha_{n+1}}{1 - \beta_{n+1}}(\|\gamma f(u_{n+1})\| + \|Aq_{n+1}\|) + \\
&\quad + \frac{\alpha_n}{1 - \beta_n}(\|\gamma f(u_n)\| + \|Aq_n\|) + \frac{1}{t_{n+1}}\int_0^{t_{n+1}} \|T(s)\rho_{n+1} - T(s)\rho_n\| ds + \\
&\quad + \|\frac{1}{t_{n+1}}\int_0^{t_{n+1}} T(s)\rho_n ds - \frac{1}{t_n}\int_0^{t_n} T(s)\rho_n ds\| - \|x_{n+1} - x_n\|.
\end{aligned} \tag{3.7}$$

Since $\rho_n = J_{M,\lambda}(I-\lambda B)\xi_n$ and $y_{n+1} = \mathscr{V}^N(x_{n+1})$, $y_n = \mathscr{V}^N(x_n)$, from the nonexpansivity of $\mathscr{V}^N$. we have

$$\begin{aligned}\|\rho_{n+1}-\rho_n\| &= \|J_{M,\lambda}(I-\lambda B)\xi_{n+1} - J_{M,\lambda}(I-\lambda B)\xi_n\| \le \\ &\le \|\xi_{n+1}-\xi_n\| = \\ &= \|J_{M,\lambda}(I-\lambda B)y_{n+1} - J_{M,\lambda}(I-\lambda B)y_n\| \le \\ &\le \|y_{n+1}-y_n\| \le \|x_{n+1}-x_n\|.\end{aligned} \tag{3.8}$$

Substituting (3.8) into (3.7), we get

$$\begin{aligned}&\|z_{n+1}-z_n\| - \|x_{n+1}-x_n\| \le \\ &\le \frac{\alpha_{n+1}}{1-\beta_{n+1}}(\|\gamma f(u_{n+1})\| + \|Aq_{n+1}\|) + \\ &\quad + \frac{\alpha_n}{1-\beta_n}(\|\gamma f(u_n)\| + \|Aq_n\|) + \frac{1}{t_{n+1}}\int_0^{t_{n+1}} \|T(s)\rho_{n+1} - T(s)\rho_n\| ds + \\ &\quad + \|\frac{1}{t_{n+1}}\int_0^{t_{n+1}} T(s)\rho_n ds - \frac{1}{t_n}\int_0^{t_n} T(s)\rho_n ds\| - \|x_{n+1}-x_n\| \le \\ &\le \frac{\alpha_{n+1}}{1-\beta_{n+1}}(\|\gamma f(u_{n+1})\| + \|Aq_{n+1}\|) + \\ &\quad + \frac{\alpha_n}{1-\beta_n}(\|\gamma f(u_n)\| + \|Aq_n\|) + \frac{1}{t_{n+1}}\int_0^{t_{n+1}} \|x_{n+1}-x_n\| ds + \\ &\quad + \|\frac{1}{t_{n+1}}\int_0^{t_{n+1}} T(s)\rho_n ds - \frac{1}{t_n}\int_0^{t_n} T(s)\rho_n ds\| - \|x_{n+1}-x_n\| \le \\ &\le \frac{\alpha_{n+1}}{1-\beta_{n+1}}(\|\gamma f(u_{n+1})\| + \|Aq_{n+1}\|) + \frac{\alpha_n}{1-\beta_n}(\|\gamma f(u_n)\| + \|Aq_n\|) + \\ &\quad + \|\frac{1}{t_{n+1}}\int_0^{t_{n+1}} T(s)\rho_n ds - \frac{1}{t_n}\int_0^{t_n} T(s)\rho_n ds\|.\end{aligned} \tag{3.9}$$

From conditions $t_n \subset (0,\infty)$ and $t_n \uparrow \infty$, we have

$$\begin{aligned}&\|\frac{1}{t_{n+1}}\int_0^{t_{n+1}} T(s)\rho_n ds - \frac{1}{t_n}\int_0^{t_n} T(s)\rho_n ds\| = \\ &= \|\frac{1}{t_{n+1}}(\int_0^{t_n} T(s)\rho_n ds + \int_{t_n}^{t_{n+1}} T(s)\rho_n ds) - \frac{1}{t_n}\int_0^{t_n} T(s)\rho_n ds\| \le \\ &\le \frac{1}{t_n t_{n+1}}\int_0^{t_n} \|(t_n - t_{n+1})T(s)\rho_n\| ds + \frac{1}{t_{n+1}}\int_{t_n}^{t_{n+1}} \|T(s)\rho_n\| ds = \\ &= \frac{t_{n+1}-t_n}{t_{n+1}}M + \frac{t_{n+1}-t_n}{t_{n+1}}M = 2M(1-\frac{t_n}{t_{n+1}}) \to 0,\end{aligned}$$

where $M = \sup_{s\ge 0,n\ge 1}\|T(s)\rho_n\|$. From (3.9) and conditions $\lim_{n\to\infty}\alpha_n = 0$ and $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$ that

$$\limsup_{n\to\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \le 0.$$

Hence, we have

$$\lim_{n\to\infty}\|z_n - x_n\| = 0.$$

Consequently

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = \lim_{n\to\infty}(1-\beta_n)\|z_n - x_n\| = 0.$$

Step 3. Next we prove that

$$\lim_{n\to\infty}\|x_n - q_n\| = 0. \tag{3.10}$$

Since

$$\begin{aligned}\|x_n - q_n\| &\le \|x_n - x_{n+1}\| + \|x_{n+1} - q_n\| \le \\ &\le \|x_n - x_{n+1}\| + \alpha_n\|\gamma f(u_n) - Aq_n\| + \beta_n\|x_n - q_n\|,\end{aligned}$$

simplifying it we have

$$\|x_n - q_n\| \le \frac{1}{1-\beta_n}\|x_n - x_{n+1}\| + \frac{\alpha_n}{1-\beta_n}\|\gamma f(u_n) - Aq_n\|.$$

Since $\alpha_n \to 0$, $\|x_{n+1} - x_n\| \to 0$, and $\{\gamma f(u_n) - Aq_n\}$ is bounded, from the condition $\lim_{n\to\infty}\alpha_n = 0$ and $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$, we have $\|x_n - q_n\| \to 0$.
Step 4. Next we prove that

$$\|x_{n+1} - T(s)x_{n+1}\| \to 0 \ (n\to\infty). \tag{3.11}$$

Since $x_{n+1} = \alpha_n\gamma f(u_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A)q_n$, then

$$\|x_{n+1} - q_n\| \le \alpha_n\|\gamma f(u_n) - Aq_n\| + \beta_n\|x_n - q_n\|.$$

From condition $\lim_{n\to\infty}\alpha_n = 0$ and $\|x_n - q_n\| \to 0$, we have

$$\|x_{n+1} - q_n\| \to 0. \tag{3.12}$$

Let $K = \{w \in C : \|w-p\| \le max\|x_1 - p\|, \frac{1}{\bar\gamma - \gamma h}\|\gamma f(p) - Ap\|$, then K is a nonempty bounded closed convex subset of C and $T(s)$-invariant. Since $\{x_n\} \subset K$ and K is bounded, there exists $r > 0$ such that $K \subset B_r$, it follows from Lemma 2.6 that

$$\lim_{n\to\infty}\|q_n - T(s)q_n\| \to 0. \tag{3.13}$$

From (3.12) and (3.13) , we have

$$\begin{aligned}\|x_{n+1} - T(s)x_{n+1}\| &= \|x_{n+1} - q_n + q_n - T(s)q_n + T(s)q_n - T(s)x_{n+1}\| \le \\ &\le \|x_{n+1} - q_n\| + \|q_n - T(s)q_n\| + \|T(s)q_n - T(s)x_{n+1}\| \le \\ &\le \|x_{n+1} - q_n\| + \|q_n - T(s)q_n\| + \|q_n - x_{n+1}\| \to 0.\end{aligned}$$

Step 5. Next we prove that

$$\begin{aligned}&(i)\ \lim_{n\to\infty}\|\mathscr{V}^{l+1}x_n-\mathscr{V}^{l}x_n\|=0,\ \forall\ l\in\{0,1,\cdots,N-1\};\\&(ii)\ \textit{Especially},\lim_{n\to\infty}\|\mathscr{V}^{N}x_n-x_n\|=\lim_{n\to\infty}\|y_n-x_n\|=0.\end{aligned}\tag{3.14}$$

In fact, for any given $p\in\mathscr{G}$ and $l\in\{0,1,\cdots,N-1\}$, Since $V_{r_{l+1}}^{\Theta_{l+1}}$ is firmly nonexpansive, we have

$$\begin{aligned}\|\mathscr{V}^{l+1}x_n-p\|^2&=\|V_{r_{l+1}}^{\Theta_{l+1}}(\mathscr{V}^{l}x_n)-V_{r_{l+1}}^{\Theta_{l+1}}p\|^2\le\\&\le\langle V_{r_{l+1}}^{\Theta_{l+1}}(\mathscr{V}^{l}x_n)-p,\ \mathscr{V}^{l}x_n-p\rangle=\\&=\langle\mathscr{V}^{l+1}x_n-p,\ \mathscr{V}^{l}x_n-p\rangle=\\&=\frac{1}{2}(\|\mathscr{V}^{l+1}x_n-p\|^2+\|\mathscr{V}^{l}x_n-p\|^2-\|\mathscr{V}^{l}x_n-\mathscr{V}^{l+1}x_n\|^2).\end{aligned}$$

It follows that

$$\|\mathscr{V}^{l+1}x_n-p\|^2\le\|x_n-p\|^2-\|\mathscr{V}^{l}x_n-\mathscr{V}^{l+1}x_n\|^2.\tag{3.15}$$

From (3.1), we have

$$\begin{aligned}&\|x_{n+1}-p\|^2=\\&=\|\alpha_n\gamma f(u_n)+\beta_nx_n+((1-\beta_n)I-\alpha_nA)q_n-p\|^2=\\&=\|\alpha_n(\gamma f(u_n)-Ap)+\beta_n(x_n-q_n)+(I-\alpha_nA)(q_n-p)\|^2\le\\&\le\|(I-\alpha_nA)(q_n-p)+\beta_n(x_n-q_n)\|^2+2\alpha_n\langle\gamma f(u_n)-Ap,x_{n+1}-p\rangle\le\\&\le[\|(I-\alpha_nA)(q_n-p)\|+\beta_n\|(x_n-q_n)\|]^2+2\alpha_n\langle\gamma f(u_n)-Ap,x_{n+1}-p\rangle\le\\&\le[(1-\alpha_n\bar\gamma)\|\rho_n-p\|+\beta_n\|x_n-q_n\|]^2+2\alpha_n\langle\gamma f(u_n)-Ap,x_{n+1}-p\rangle=\\&=(1-\alpha_n\bar\gamma)^2\|\rho_n-p\|^2+\beta_n^2\|x_n-q_n\|^2+2(1-\alpha_n\bar\gamma)\beta_n\|\rho_n-p\|\cdot\|x_n-q_n\|+\\&\quad+2\alpha_n\|\gamma f(u_n)-Ap\|\cdot\|x_{n+1}-p\|.\end{aligned}\tag{3.16}$$

Since

$$\|\rho_n-p\|\le\|\xi_n-p\|\le\|\mathscr{V}^{N}x_n-p\|\le\|\mathscr{V}^{l+1}x_n-p\|\ \forall\ l\in\{0,1,\cdots,N-1\}.$$

Substituting (3.15) into (3.16), it yields

$$\begin{aligned}&\|x_{n+1}-p\|^2\le\\&\le(1-\alpha_n\bar\gamma)^2\{\|x_n-p\|^2-\|\mathscr{V}^{l}x_n-\mathscr{V}^{l+1}x_n\|^2\}+\beta_n^2\|x_n-q_n\|^2+\\&\quad+2(1-\alpha_n\bar\gamma)\cdot\beta_n\|\rho_n-p\|\cdot\|x_n-q_n\|+2\alpha_n\|\gamma f(u_n)-Ap\|\cdot\|x_{n+1}-p\|=\\&=(1-2\alpha_n\bar\gamma+(\alpha_n\bar\gamma)^2)\|x_n-p\|^2-(1-\alpha_n\bar\gamma)^2\|\mathscr{V}^{l}x_n-\mathscr{V}^{l+1}x_n\|^2+\beta_n^2\|x_n-q_n\|^2+\\&\quad+2(1-\alpha_n\bar\gamma)\beta_n\|\rho_n-p\|\cdot\|x_n-q_n\|+2\alpha_n\|\gamma f(u_n)-Ap\|\cdot\|x_{n+1}-p\|.\end{aligned}$$

Simplifying it we have

$$\begin{aligned}&(1-\alpha_n\bar{\gamma})^2\|\mathscr{V}^l x_n-\mathscr{V}^{l+1}x_n\|^2\le\\&\le\|x_n-p\|^2-\|x_{n+1}-p\|^2+\alpha_n\bar{\gamma}^2\|x_n-p\|^2+\\&\quad+\beta_n^2\|x_n-q_n\|^2+2(1-\alpha_n\bar{\gamma})\beta_n\|\rho_n-p\|\cdot\|x_n-q_n\|+\\&\quad+2\alpha_n\|\gamma f(u_n)-Ap\|\cdot\|x_{n+1}-p\|.\end{aligned}$$

Since $\alpha_n\to 0$, $\|x_{n+1}-x_n\|\to 0$, $\|x_n-q_n\|\to 0$, it yields $\|\mathscr{V}^l x_n-\mathscr{V}^{l+1}x_n\|\to 0$.
Step 6. Now we prove that for any given $p\in\mathscr{G}$

$$\lim_{n\to\infty}\|By_n-Bp\|=0. \tag{3.17}$$

In fact, it follows from (3.3) that

$$\begin{aligned}\|\rho_n-p\|^2\le\|\xi_n-p\|^2&=\|J_{M,\lambda}(I-\lambda B)y_n-J_{M,\lambda}(I-\lambda B)p\|^2\le\\&\le\|(I-\lambda B)y_n-(I-\lambda B)p\|^2=\\&=\|y_n-p\|^2-2\lambda\langle y_n-p,By_n-Bp\rangle+\lambda^2\|By_n-Bp\|^2\le\\&\le\|y_n-p\|^2+\lambda(\lambda-2\alpha)\|By_n-Bp\|^2\le\\&\le\|x_n-p\|^2+\lambda(\lambda-2\alpha)\|By_n-Bp\|^2.\end{aligned}\tag{3.18}$$

Substituting (3.18) into (3.16), we obtain

$$\begin{aligned}&\|x_{n+1}-p\|^2\le\\&\le(1-\alpha_n\bar{\gamma})^2\{\|x_n-p\|^2+\lambda(\lambda-2\alpha)\|By_n-Bp\|^2\}+\beta_n^2\|x_n-q_n\|^2+\\&+2(1-\alpha_n\bar{\gamma})\beta_n\|\rho_n-p\|\cdot\|x_n-q_n\|+2\alpha_n\|\gamma f(u_n)-Ap\|\cdot\|x_{n+1}-p\|.\end{aligned}$$

Simplifying it, we have

$$\begin{aligned}&(1-\alpha_n\bar{\gamma})^2\lambda(2\alpha-\lambda)\|By_n-Bp\|^2\le\\&\le\|x_n-p\|^2-\|x_{n+1}-p\|^2+\alpha_n\bar{\gamma}^2\|x_n-p\|^2+\beta_n^2\|x_n-q_n\|^2+\\&\quad+2(1-\alpha_n\bar{\gamma})\beta_n\|\rho_n-p\|\cdot\|x_n-q_n\|+2\alpha_n\|\gamma f(u_n)-Ap\|\cdot\|x_{n+1}-p\|.\end{aligned}$$

Since $\alpha_n\to 0$, $0<\liminf_{n\to\infty}\beta_n\le\limsup_{n\to\infty}\beta_n<1$, $\|x_{n+1}-x_n\|\to 0$, $\|x_n-q_n\|\to 0$, and $\{\gamma f(u_n)-Ap\},\{x_n\}$ are bounded, these imply that $\|By_n-Bp\|\to 0$ $(n\to\infty)$.
Step 7. Next we prove that

$$\lim_{n\to\infty}\|y_n-\rho_n\|=0. \tag{3.19}$$

In fact, since

$$\|y_n-\rho_n\|\le\|y_n-\xi_n\|+\|\xi_n-\rho_n\|,$$

for the purpose, it is sufficient to prove

$$\|y_n-\xi_n\|\to 0 \ \textit{and}\ \|\xi_n-\rho_n\|\to 0.$$

(a) First we prove that $\|y_n - \xi_n\| \to 0$.

In fact, since

$$\begin{aligned}
&\|\xi_n - p\|^2 = \\
&= \|J_{M,\lambda}(I - \lambda B)y_n - J_{M,\lambda}(I - \lambda B)p\|^2 \leq \\
&\leq \langle y_n - \lambda B y_n - (p - \lambda B p), \xi_n - p\rangle = \\
&= \frac{1}{2}\{\|y_n - \lambda B y_n - (p - \lambda B p)\|^2 + \|\xi_n - p\|^2 - \|y_n - \lambda B y_n - (p - \lambda B p) - (\xi_n - p)\|^2\} \leq \\
&\leq \frac{1}{2}\{\|y_n - p\|^2 + \|\xi_n - p\|^2 - \|y_n - \xi_n - \lambda(B y_n - B p)\|^2\} \leq \\
&\leq \frac{1}{2}\{\|y_n - p\|^2 + \|\xi_n - p\|^2 - \|y_n - \xi_n\|^2 + 2\lambda\langle y_n - \xi_n, B y_n - B p\rangle - \lambda^2\|B y_n - B p\|^2\}
\end{aligned}$$

we have

$$\|\xi_n - p\|^2 \leq \|y_n - p\|^2 - \|y_n - \xi_n\|^2 + 2\lambda\langle y_n - \xi_n, B y_n - B p\rangle - \lambda^2\|B y_n - B p\|^2. \quad (3.20)$$

Substituting (3.20) into (3.16), it yields that

$$\begin{aligned}
\|x_{n+1} - p\|^2 \leq (1 - \alpha_n\bar{\gamma})^2\{&\|y_n - p\|^2 - \|y_n - \xi_n\|^2 + \\
&+ 2\lambda\langle y_n - \xi_n, B y_n - B p\rangle - \lambda^2\|B y_n - B p\|^2\} + \beta_n^2\|x_n - q_n\|^2 + \\
&+ 2(1 - \alpha_n\bar{\gamma})\beta_n\|\rho_n - p\| \cdot \|x_n - q_n\| + 2\alpha_n\|\gamma f(u_n) - Ap\| \cdot \|x_{n+1} - p\|.
\end{aligned}$$

Simplifying it we have

$$\begin{aligned}
&(1 - \alpha_n\bar{\gamma})^2\|y_n - \xi_n\|^2 \leq \\
&\leq (\|x_n - x_{n+1}\|) \cdot (\|x_n - p\| + \|x_{n+1} - p\|) + \alpha_n\bar{\gamma}^2\|x_n - p\|^2 + \\
&+ 2(1 - \alpha_n\bar{\gamma}^2)\lambda\langle y_n - \xi_n, B y_n - B p\rangle - (1 - \alpha_n\bar{\gamma})^2\lambda^2\|B y_n - B p\|^2 + \beta_n^2\|x_n - q_n\|^2 + \\
&+ 2(1 - \alpha_n\bar{\gamma})\beta_n\|\rho_n - p\| \cdot \|x_n - q_n\| + 2\alpha_n\|\gamma f(u_n) - Ap\| \cdot \|x_{n+1} - p\|.
\end{aligned}$$

Since $\alpha_n \to 0$, $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$, $\|x_n - q_n\| \to 0$, $\|B y_n - B p\| \to 0$ $(n \to \infty)$, $\|x_{n+1} - x_n\| \to 0$ and $\{\gamma f(u_n) - Ap\}, \{x_n\}, \{\rho_n\}$ are bounded, these imply that $\|y_n - \xi_n\| \to 0$ $(n \to \infty)$.

(b) Next we prove that

$$\lim_{n\to\infty}\|\xi_n - \rho_n\| = 0. \quad (3.21)$$

In fact, since $\|\xi_n - \rho_n\| = \|J_{M,\lambda}(I - \lambda B)y_n - J_{M,\lambda}(I - \lambda B)\xi_n\| \leq \|y_n - \xi_n\| \to 0$, and so $\|y_n - \rho_n\| = \|y_n - \xi_n + \xi_n - \rho_n\| \leq \|y_n - \xi_n\| + \|\xi_n - \rho_n\| \to 0$.

Step 8. Next we prove that

$$\limsup_{n\to\infty}\langle\gamma f(x^*) - Ax^*, x_{n+1} - x^*\rangle \leq 0. \quad (3.22)$$

(a) First, we prove that

$$\limsup_{n\to\infty}\langle\frac{1}{t_n}\int_0^{t_n} T(s)\rho_n ds - x^*, \gamma f(x^*) - Ax^*\rangle \leq 0. \quad (3.23)$$

To see this, there exist a subsequence $\{\rho_{n_i}\}$ of $\{\rho_n\}$ such that

$$\limsup_{n\to\infty}\langle\frac{1}{t_n}\int_0^{t_n} T(s)\rho_n ds - x^*, \gamma f(x^*) - Ax^*\rangle =$$
$$= \limsup_{i\to\infty}\langle\frac{1}{t_{n_i}}\int_0^{t_{n_i}} T(s)\rho_{n_i} ds - x^*, \gamma f(x^*) - Ax^*\rangle$$

we may also assume that $\rho_{n_i} \rightharpoonup w$, then $q_{n_i} = \frac{1}{t_{n_i}}\int_0^{t_{n_i}} T(s)\rho_{n_i} ds \rightharpoonup w$. Since $\|x_n - q_n\| \to 0$, we have $x_{n_i} \rightharpoonup w$.

Next, we prove that

$$w \in \mathscr{G}.$$

(1^0) We first prove that $w \in F(\mathscr{T})$. In fact, since $\{x_{n_i}\} \rightharpoonup w$. From Lemma 2.7 and Step 4, we obtain $w \in F(\mathscr{T})$.

(2^0) Now we prove that $w \in \cap_{l=1}^N MEP(\Theta_l, \varphi_l)$.

Since $x_{n_i} \rightharpoonup w$ and noting Step 5, without loss of generality, we may assume that $\mathscr{V}^l x_{n_i} \rightharpoonup w$, $\forall\ l \in \{0, 1, 2, \cdots, N-1\}$. Hence for any $x \in C$ and for any $l \in \{0, 1, 2, \cdots, N-1\}$, we have

$$\langle\frac{K'_{l+1}(\mathscr{V}^{l+1}x_{n_i}) - K'_{l+1}(\mathscr{V}^l x_{n_i})}{r_{l+1}},\ \eta_{l+1}(x, \mathscr{V}^{l+1}x_{n_i})\rangle \geq$$
$$\geq -\Theta_{l+1}(\mathscr{V}^{l+1}x_{n_i}, x) - \varphi_{l+1}(x) + \varphi_{l+1}(\mathscr{V}^{l+1}x_{n_i}).$$

By the assumptions and by the condition (H_2) we know that the function φ_i and the mapping $x \mapsto (-\Theta_{l+1}(x, y))$ both are convex and lower semi-continuous, hence they are weakly lower semi-continuous. These together with $\frac{K'_{l+1}(\mathscr{V}^{l+1}x_{n_i}) - K'_{l+1}(\mathscr{V}^l x_{n_i})}{r_{l+1}} \to 0$ and $\mathscr{V}^{l+1}x_{n_i} \rightharpoonup w$, we have

$$0 = \liminf_{i\to\infty}\{\langle\frac{K'_{l+1}(\mathscr{V}^{l+1}x_{n_i}) - K'_{l+1}(\mathscr{V}^l x_{n_i})}{r_{l+1}}, \eta_{l+1}(x, \mathscr{V}^{l+1}x_{n_i})\rangle\} \geq$$
$$\geq \liminf_{i\to\infty}\{-\Theta_{l+1}(\mathscr{V}^{l+1}x_{n_i}, x) - \varphi_{l+1}(x) + \varphi_{l+1}(\mathscr{V}^{l+1}x_{n_i})\}.$$

i.e.,

$$\Theta_{l+1}(w, x) + \varphi_{l+1}(x) - \varphi_{l+1}(w) \geq 0$$

for all $x \in C$ and $l \in \{0, 1, \cdots, N-1\}$, hence $w \in \cap_{l=1}^N MEP(\Theta_l, \varphi_l)$.

(3^0) Now we prove that $w \in VI(H, B, M)$.

In fact, since B is α-inverse-strongly monotone, it follows from Proposition 1.1 that B is a $\frac{1}{\alpha}$-Lipschitz continuous monotone mapping and $D(B) = H$ (where $D(B)$ is the domain of B). It follows from Lemma 2.4 that $M + B$ is maximal monotone. Let $(\nu, g) \in Graph(M + B)$, i.e., $g - B\nu \in M(\nu)$. Since $x_{n_i} \rightharpoonup w$ and noting Step 5, without loss of generality, we may assume that $\mathscr{V}^l x_{n_i} \rightharpoonup w$, in particular, we have

$y_{n_i} = \mathscr{V}^N x_{n_i} \rightharpoonup w$. From $\|y_n - \rho_n\| \to 0$, we can prove that $\rho_{n_i} \rightharpoonup w$. Again since $\rho_{n_i} = J_{M,\lambda}(I - \lambda B)\xi_{n_i}$, we have

$$\xi_{n_i} - \lambda B\xi_{n_i} \in (I + \lambda M)\rho_{n_i},\ i.e.,\ \frac{1}{\lambda}(\xi_{n_i} - \rho_{n_i} - \lambda B\xi_{n_i}) \in M(\rho_{n_i}).$$

By virtue of the maximal monotonicity of M, we have

$$\langle \nu - \rho_{n_i}, g - B\nu - \frac{1}{\lambda}(\xi_{n_i} - \rho_{n_i} - \lambda B\xi_{n_i})\rangle \geq 0$$

and so

$$\begin{aligned}\langle \nu - \rho_{n_i}, g\rangle &\geq \langle \nu - \rho_{n_i}, B\nu + \frac{1}{\lambda}(\xi_{n_i} - \rho_{n_i} - \lambda B\xi_{n_i})\rangle = \\ &= \langle \nu - \rho_{n_i}, B\nu - B\rho_{n_i} + B\rho_{n_i} - B\xi_{n_i} + \frac{1}{\lambda}(\xi_{n_i} - \rho_{n_i})\rangle \geq \\ &\geq 0 + \langle \nu - \rho_{n_i}, B\rho_{n_i} - B\xi_{n_i}\rangle + \langle \nu - \rho_{n_i}, \frac{1}{\lambda}(\xi_{n_i} - \rho_{n_i})\rangle.\end{aligned}$$

Since $\|\xi_n - \rho_n\| \to 0$, $\|B\xi_n - B\rho_n\| \to 0$ and $\rho_{n_i} \rightharpoonup w$, we have

$$\lim_{n_i\to\infty} \langle \nu - \rho_{n_i}, g\rangle = \langle \nu - w, g\rangle \geq 0.$$

Since $M + B$ is maximal monotone, this implies that $\theta \in (M + B)(w)$, i.e., $w \in VI(H, B, M)$, and so $w \in \mathscr{G}$.

Since $x^* = P_{\mathscr{G}}(I - A + \gamma f)(x^*)$, we have

$$\begin{aligned}&\limsup_{n\to\infty} \langle \frac{1}{t_n} \int_0^{t_n} T(s)\rho_n ds - x^*, \gamma f(x^*) - Ax^*\rangle = \\ &= \limsup_{i\to\infty} \langle \frac{1}{t_{n_i}} \int_0^{t_{n_i}} T(s)\rho_{n_i} ds - x^*, \gamma f(x^*) - Ax^*\rangle = \\ &= \limsup_{i\to\infty} \langle q_{n_i} - x^*, \gamma f(x^*) - Ax^*\rangle = \\ &= \langle w - x^*, \gamma f(x^*) - Ax^*\rangle \leq 0.\end{aligned}$$

(b) Now we prove that

$$\limsup_{n\to\infty} \langle \gamma f(x^*) - Ax^*, x_{n+1} - x^*\rangle \leq 0.$$

From $\|x_{n+1} - q_n\| \to 0$ and (a), we have

$$\begin{aligned}&\limsup_{n\to\infty} \langle \gamma f(x^*) - Ax^*, x_{n+1} - x^*\rangle = \\ &= \limsup_{n\to\infty} \langle \gamma f(x^*) - Ax^*, x_{n+1} - q_n + q_n - x^*\rangle \leq \\ &\leq \limsup_{n\to\infty} \langle \gamma f(x^*) - Ax^*, x_{n+1} - q_n\rangle + \limsup_{n\to\infty} \langle \gamma f(x^*) - Ax^*, q_n - x^*\rangle \leq 0.\end{aligned}$$

Step 9. Finally we prove that

$$x_n \to x^*.$$

Indeed, from (3.1) , we have

$$
\begin{aligned}
&\|x_{n+1}-x^*\|^2 = \\
&= \|\alpha_n(\gamma f(u_n)-Ax^*)+\beta_n(x_n-x^*)+((1-\beta_n)I-\alpha_n A)(q_n-x^*)\|^2 \leq \\
&\leq \|\beta_n(x_n-x^*)+((1-\beta_n)I-\alpha_n A)(q_n-x^*)\|^2+2\alpha_n\langle\gamma f(u_n)-Ax^*,x_{n+1}-x^*\rangle \leq \\
&\leq [\|((1-\beta_n)I-\alpha_n A)(q_n-x^*)\|+\beta_n\|x_n-x^*\|]^2+ \\
&\quad +2\alpha_n\gamma\langle f(u_n)-f(x^*),x_{n+1}-x^*\rangle+2\alpha_n\langle\gamma f(x^*)-Ax^*,x_{n+1}-x^*\rangle \leq \\
&\leq [\|(1-\beta_n-\alpha_n\bar{\gamma})\|\rho_n-x^*\|+\beta_n\|x_n-x^*\|]^2+2\alpha_n\gamma h\|x_n-x^*\|\cdot\|x_{n+1}-x^*\|+ \\
&\quad +2\alpha_n\langle\gamma f(x^*)-Ax^*,x_{n+1}-x^*\rangle \leq \\
&\leq (1-\alpha_n\bar{\gamma})^2\|x_n-x^*\|^2+\alpha_n\gamma h\{\|x_n-x^*\|^2+\|x_{n+1}-x^*\|^2\}+ \\
&+2\alpha_n\langle\gamma f(x^*)-Ax^*,x_{n+1}-x^*\rangle.
\end{aligned}
$$

This implies that

$$
\begin{aligned}
&\|x_{n+1}-x^*\|^2 \leq \\
&\leq \frac{(1-\alpha_n\bar{\gamma})^2+\alpha_n\gamma h}{1-\alpha_n\gamma h}\|x_n-x^*\|^2+\frac{2\alpha_n}{1-\alpha_n\gamma h}\langle\gamma f(x^*)-Ax^*,x_{n+1}-x^*\rangle = \\
&= [1-\frac{2(\bar{\gamma}-\gamma h)\alpha_n}{1-\alpha_n\gamma h}]\|x_n-x^*\|^2+\frac{(\alpha_n\bar{\gamma})^2}{1-\alpha_n\gamma h}\|x_n-x^*\|^2+\frac{2\alpha_n}{1-\alpha_n\gamma h}\langle\gamma f(x^*)- \\
&\quad -Ax^*,x_{n+1}-x^*\rangle \leq \\
&\leq [1-\frac{2(\bar{\gamma}-\gamma h)\alpha_n}{1-\alpha_n\gamma h}]\|x_n-x^*\|^2+\frac{2(\bar{\gamma}-\gamma h)\alpha_n}{1-\alpha_n\gamma h}\{\frac{\alpha_n\bar{\gamma}^2}{2(\bar{\gamma}-\gamma h)}\|x_n-x^*\|^2+ \\
&\quad +\frac{1}{\bar{\gamma}-\gamma h}\langle\gamma f(x^*)-Ax^*,x_{n+1}-x^*\rangle\} = \\
&= (1-l_n)\|x_n-x^*\|^2+\delta_n,
\end{aligned}
$$

where

$$l_n=\frac{2(\bar{\gamma}-\gamma h)\alpha_n}{1-\alpha_n\gamma h},$$

and

$$\delta_n=\frac{2(\bar{\gamma}-\gamma h)\alpha_n}{1-\alpha_n\gamma h}\{\frac{\alpha_n\bar{\gamma}^2}{2(\bar{\gamma}-\gamma h)}\|x_n-x^*\|^2+\frac{1}{\bar{\gamma}-\gamma h}\langle\gamma f(x^*)-Ax^*,\ x_{n+1}-x^*\rangle\}.$$

It is easy to see that $l_n \to 0$, $\sum_{n=1}^{\infty} l_n = \infty$ and $\limsup_{n\to\infty}\frac{\delta_n}{l_n} \leq 0$. Hence the sequence $\{x_n\}$ converges strongly to x^*.

This completes the proof of Theorem 3.2. □

Corollary 3.3. *Let H, C, A, B, M, f, $\mathscr{T}$, $\mathscr{F}$, φ_i, $\eta_i, K_i (i = 1, 2, \cdots, N)$ be the same as Theorem 3.2. Let $\{x_n\}$, $\{\rho_n\}$, $\{\xi_n\}$ and $\{y_n\}$ be explicit iterative sequences generated by $x_1 \in H$ and*

$$\begin{cases} x_{n+1} = \alpha_n \gamma f(\frac{1}{t_n}\int_0^{t_n} T(s)x_n ds) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A)\frac{1}{t_n}\int_0^{t_n} T(s)\rho_n ds, \\ \rho_n = P_C(I - \lambda B)\xi_n, \\ \xi_n = P_C(I - \lambda B)y_n, \\ y_n = V_{r_N}^{\Theta_N} \dots V_{r_2}^{\Theta_2} V_{r_1}^{\Theta_1} x_n, \end{cases} \quad \forall n \geq 1, \tag{3.24}$$

where $r_i (i = 1, 2, \cdots, N)$ are a finite family of positive numbers, $\lambda \in (0, 2\alpha]$, $\{\alpha_n\}$, $\{\beta_n\} \subset [0, 1]$ and $\{t_n\} \subset (0, \infty)$ is a sequence with $t_n \uparrow \infty$. If $\mathscr{G} := F(\mathscr{T}) \bigcap MEP(\mathscr{F}) \bigcap VI(C, B) \neq \emptyset$ and the following conditions are satisfied:

(i) *for each $x \in C$, there exist a bounded subset $D_x \subseteq C$ and $z_x \in C$ such that for any $y \in C \setminus D_x$,*

$$\Theta_i(y, z_x) + \varphi_i(z_x) - \varphi_i(y) + \frac{1}{r_i}\langle K_i'(y) - K_i'(x), \eta_i(z_x, y)\rangle < 0,$$

(ii) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$,

then the sequence $\{x_n\}$ converges strongly to some point $x^* = P_{\mathscr{G}}(I - A + \gamma f)(x^*)$, provided that $V_{r_i}^{\Theta_i}$ is firmly nonexpansive.

Proof. Taking $M = \partial\delta_C : H \to 2^H$ in Theorem 3.2, where $\delta_C : H \to [0, \infty)$ is the indicator function of C, i.e.,

$$\delta_C = \begin{cases} 0, & x \in C, \\ +\infty, & x \notin C, \end{cases}$$

then the variational inclusion problem (1.2) is equivalent to variational inequality (1.3), i.e., to find $u \in C$ such that

$$\langle B(u), v - u\rangle \geq 0, \forall v \in C.$$

Again, since $M = \partial\delta_C$, the restriction of $J_{M,\lambda}$ on C is an identity mapping, i.e., $J_{M,\lambda}|_C = I$ and so we have

$$P_C(I - \lambda B)k_n = J_{M,\lambda}(P_C(I - \lambda B)k_n); \quad P_C(I - \lambda B)y_n = J_{M,\lambda}(P_C(I - \lambda B)y_n).$$

Hence the conclusion of Corollary 3.3 can be obtained form Theorem 3.2 immediately. □

REFERENCES

[1] L.C. Ceng, J.C. Yao, *A hybrid iterative scheme for mixed equilibrium problems and fixed point problems*, J. Comput. Appl. Math. **214** (2008), 186–201.

[2] S. Chang, *Set-valued variational inclusions in Banach space*, J. Math. Anal. Appl. **248** (2000), 438–454.

[3] S. Chang, *Existence and approximation of solutions of Set-valued variational inclusions in Banach space*, Nonlinear Anal. **47** (2001) 1, 583–594.

[4] S. Chang, H.W. Joseph Lee, C. Chan, *A new method for solving equilibrium problem fixed point problem and variational inequality problem with application to optimization*, Nonlinear Anal. **70** (2009), 3307–3319.

[5] V.F. Demyanov, G.E. Stavroulakis, L.N. Polyakova, P.D. Panagiotopoulos, *Quasi-differentiability and Nonsmooth Modeling in Mechanics. Engineering and Economics*, Kluwer Academic, Dordrecht, 1996.

[6] S. Li, L. Li, Y. Su, *General iterative methods for a one-parameter nonexpansive semigroup in Hilbert space*, Nonlinear Anal., Theory Methods Appl. 70, No. 9 (A), 2065–2071 (2009), doi:10.1016/j.na.2008.04.007.

[7] J.L. Lions, G. Stampacchia, *Variational inequalities*, Comm. Pure. Appl. Math. **20** (1967), 493–517.

[8] D. Pascali, *Nolinear Mappings of Monotone Type*, Sijthoff and Noordhoff International Publishers, The Netherlands, 1978.

[9] S. Saeidi, *Iterative algorithms for finding common solutions of variational inequalities and systems of equilibrium problems and fixed points of families and semigroups of nonexpansive mappings*, Nonlinear Anal. (2008), doi:10.1016/j.na.2008.09.009.

[10] T. Shimizu, W. Takahashi, *Strong convergence to common fixed points of families of nonexpansive mappings*, J. Math. Anal. Appl. **211** (1997), 71–83.

[11] S. Takahashi, W. Takahshi, *Viscosity approximation methods for equilibrium problems and fixed point problems in Hilbert space*, J. Math. Anal. Appl. **331** (2007), 506–515.

[12] H.K. Xu, *An iterative approach to quadratic optimization*, J. Optim. Theory Appl. **116** (2003) 3, 659–678.

[13] S. Zhang, J.H. Joseph Lee, C.K. Chan, *Algorithms of common solutions for quasi-variational inclusion and fixed point problems*, Appl. Math. Mech. **29** (2008), 1–11.

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