Opuscula Mathematica • Vol. 30 • No. 4 • 2010

ON SOME IMPULSIVE FRACTIONAL DIFFERENTIAL EQUATIONS IN BANACH SPACES

JinRong Wang, W. Wei, YanLong Yang

Abstract. This paper deals with some impulsive fractional differential equations in Banach spaces. Utilizing the Leray-Schauder fixed point theorem and the impulsive nonlinear singular version of the Gronwall inequality, the existence of PC-mild solutions for some fractional differential equations with impulses are obtained under some easily checked conditions. At last, an example is given for demonstration.

Keywords: fractional differential equations with impulses, nonlinear impulsive singular version of the Gronwall inequality, PC-mild solutions, existence.

Mathematics Subject Classification: 45N05, 93C25.

1. INTRODUCTION

During the past decades, impulsive differential equations have attracted much interest since it is much richer than the corresponding theory of differential equations (see for instance [14, 25, 54] and references therein). Recently, impulsive evolution equations and their optimal control problems in infinite dimensional spaces have been investigated by many authors including Ahmed, Benchohra, Ntouyas, Liu, Nieto and us (see for instance [1–5, 8, 9, 29], [38, 39, 50–53] and references therein). Specially, we also studied the impulsive periodic system in infinite dimensional spaces (see [43–47]).

On the other hand, the fractional differential equations have recently been proved to be valuable tools in the modeling of many phenomena in various fields of engineering, physics, economics and science. We can find numerous applications in viscoelasticity, electrochemistry, control, porous media, electromagnetic, etc. [16, 19–21, 33, 37]. In recent years, there has been a significant development in fractional differential equations. One can see the monographs of Kilbas *et al.* [24], Miller and Ross [31], Podlubny [41], Lakshmikantham *et al.* [28], and the papers [6, 7, 10, 15–19, 22, 24, 26, 27, 34, 35] and the references therein.

However, to our knowledge, the theory for impulsive fractional differential equations in Banach spaces has not yet been sufficiently developed. Very recently, Benchohra *et al.* [10, 12] applied the Banach contraction principle, Schaefer's fixed point theorem and the nonlinear alternative of the Leray-Schauder type or measure of noncompactness to a class of impulsive fractional differential equations without unbounded operator. A class of initial value problem for impulsive fractional differential equations with variable times is also considered in [13]. Balachandran *et al.* [13] using fractional calculus and fixed point theorems for a class of impulsive fractional evolution equations with bounded time-varying linear operator. Mophou *et al.* [36], Wang *et al.* [49], apply semigroup theory and fixed point theorems to study the impulsive fractional differential equations with an unbounded operator in Banach spaces.

Motivated by the above work including [34–36, 48, 49], the main purpose of this paper is to consider the following fractional differential equations with impulses

$$\begin{cases} D_t^{\alpha}x(t) = Ax(t) + t^n f(t, x(t)), \ \alpha \in (0,1], \quad n \in Z^+, \ t \in J = [0,b], \quad t \neq t_k, \\ x(0) = x_0, \\ \Delta x(t_k) = I_k(x(t_k)) = x(t_k^+) - x(t_k^-), \ k = 1, 2, \dots, \delta, \ 0 < t_1 < t_2 < \dots < t_\delta < b, \end{cases} \tag{1.1}$$

where A: $D(A) \subset X \to X$ is the generator of a C_0-semigroup $\{T(t), t \geq 0\}$ on a Banach space X, D_t^{α} is the Caputo fractional derivative, f: $J \times X \to X$ is specified later, x_0 is an element of X, I_k: $X \to X$ is a nonlinear map which determines the size of the jump at t_k, $0 = t_0 < t_1 < t_2 < \dots < t_\delta < t_{\delta+1} = b$, $x(t_k^+) = \lim_{h\to 0^+} = x(t_k+h)$ and $x(t_k^-) = x(t_k)$ represents respectively the right and left limits of $x(t)$ at $t = t_k$.

In order to obtain the existence of solutions for impulsive fractional differential equations, some authors use Krasnoselskii's fixed point theorem or contraction mapping principle. It is obvious that the conditions for Krasnoselskii's fixed point theorem are not easily verified sometimes and the conditions for the contraction mapping principle are too strong. Some authors give the prior estimate of the solutions for impulsive fractional differential equations, however, the condition on f is a little strong.

Here, we use the Leray-Schauder fixed point theorem to obtain the existence of PC-mild solutions for system (1.1) under some easily checked conditions. First, we construct an operator H for system (1.1), then use a generalized Ascoli-Arzela theorem (see Theorem 2.5) and overcome some difficulties to show the compactness of H which is very important. With the help of an impulsive nonlinear singular version of the Gronwall inequality (see Theorem 2.7), the key estimate of the fixed point set $\{x = \sigma Hx, \sigma \in [0,1]\}$ can be established successfully. Therefore, the existence of PC-mild solutions for system (1.1) is shown. Our methods are different from previous work and we give a new way to show the existence of solutions for impulsive fractional differential equations.

The paper is organized as follows. In Section 2, we introduce the PC-mild solution of system (1.1) and recall some basis results including the impulsive nonlinear singular version of the Gronwall inequality. In Section 3, the existence of PC-mild solutions for system (1.1) is proved under some easily checked conditions. Finally, an example is given to demonstrate the applicability of our result.

2. PRELIMINARIES

Let $\pounds_b(X)$ be the Banach space of all linear and bounded operators on X. For a C_0-semigroup $\{T(t), t \geq 0\}$ on X, we set $M \equiv \sup_{t\in J} \|T(t)\|_{\pounds_b(X)}$. Let $C(J,X)$ be the Banach space of all X-valued continuous functions from $J = [0,b]$ into X endowed with the norm $\|x\|_C = \sup_{t\in J} \|x(t)\|$. We also introduce the set of functions $PC(J,X) \equiv \big\{x : J \to X \mid x \text{ is continuous at } t \in J\backslash\{t_1, t_2, \dots, t_\delta\}, \text{ and } x \text{ is continuous from left and has right hand limits at } t \in \{t_1, t_2, \dots, t_\delta\}\big\}$. Endowed with the norm

$$\|x\|_{PC} = \max\left\{\sup_{t\in J}\|x(t+0)\|, \sup_{t\in J}\|x(t-0)\|\right\},$$

$(PC(J,X), \|\cdot\|_{PC})$ is a Banach space.

Let us recall the following definitions. For more details see [41].

Definition 2.1. A real function $f(t)$ is said to be in the space C_α, $\alpha \in R$ if there exists a real number $\kappa > \alpha$, such that $f(t) = t^\kappa g(t)$, where $g \in C[0,\infty)$ and it is said to be in the space C_α^m iff $f^{(m)} \in C_\alpha$, $m \in N$.

Definition 2.2. The Riemann-Liouville fractional integral operator of order $\alpha > 0$ of a function $f \in C_\alpha$, $\alpha \geq -1$ is defined as

$$I_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s)ds,$$

where $\Gamma(\cdot)$ is the Euler gamma function.

Definition 2.3. If the function $f \in C_{-1}^\zeta$, $\zeta \in N$, the fractional derivative of order $\alpha > 0$ of a function $f(t)$ in the Caputo sense is given by

$$\frac{d^\alpha f(t)}{dt^\alpha} = \frac{1}{\Gamma(\zeta-\alpha)}\int_0^t (t-s)^{\zeta-\alpha-1} f^{(\zeta)}(s)ds,\ \zeta - 1 < \alpha \leq \zeta.$$

Based on [36] (Definition 3.2 and Lemma 3.3), we use the following definition of a PC-mild solution for system (1.1).

Definition 2.4. By a PC-mild solution of the system (1.1) we mean the function $x \in PC(J,X)$ which satisfies

$$\begin{aligned} x(t) = T(t)x_0 &+ \frac{1}{\Gamma(\alpha)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}T(t-s)s^n f(s,x(s))\,ds+ \\ &+ \frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}T(t-s)s^n f(s,x(s))\,ds+ \\ &+ \sum_{0<t_k<t} T(t-t_k)I_k(x(t_k)). \end{aligned} \tag{2.1}$$

The following results will be used later.

Lemma 2.5 (Generalized Ascoli-Arzela theorem, Theorem 2.1, [50]). *Suppose $\mathcal{W} \subset PC(J, X)$ be a subset. If the following conditions are satisfied:*

(1) *$\mathcal{W}$ is a uniformly bounded subset of $PC(J, X)$.*
(2) *$\mathcal{W}$ is equicontinuous in (t_k, t_{k+1}), $k = 0, 1, 2, \ldots, \delta$, where $t_0 = 0$, $t_{\delta+1} = b$.*
(3) *$\mathcal{W}(t) \equiv \{x(t) \mid x \in \mathcal{W}, t \in J \backslash \{t_1, \ldots, t_\delta\}\}$, $\mathcal{W}(t_k + 0) \equiv \{x(t_k + 0) \mid x \in \mathcal{W}\}$ and $\mathcal{W}(t_k - 0) \equiv \{x(t_k - 0) \mid x \in \mathcal{W}\}$ are relatively compact subsets of $PC(J, X)$.*

Then $\mathcal{W}$ is a relatively compact subset of $PC(J, X)$.

Lemma 2.6 (Lemma 2.1, [42]). *For all $\beta > 0$ and $\vartheta > -1$,*

$$\int_0^t (t-s)^{\beta-1} s^{\vartheta} ds = C(\beta, \vartheta) t^{\beta+\vartheta},$$

where

$$C(\beta, \vartheta) = \frac{\Gamma(\beta)\Gamma(\vartheta+1)}{\Gamma(\beta+\vartheta+1)}.$$

Lemma 2.7 (Impulsive nonlinear singular version of the Gronwall inequality, Theorem 3.1, [42]). *Let $x \in PC([0, \infty), X)$ and satisfy the following inequality*

$$x(t) \le a(t) + b(t) \int_0^t (t-s)^{\alpha-1} s^{\gamma} F_1(s) x^m(s) ds + d(t) \sum_{0<t_k<t} \eta_k x(t_k), t \ge 0, \tag{2.2}$$

where $a(t)$, $b(t)$, $d(t)$ and $F_1(t)$ are nonnegative continuous functions, $\eta_k \ge 0$ are constants.

(1) *If $\frac{1}{2} \ge \alpha > 0$, $-\frac{1}{2} \ge \gamma > -1$, then it holds that for $t \in (t_k, t_{k+1}]$,*

$$\begin{aligned} x(t) \le & \left[(k+3)^{q-1} f_p(t) \prod_{l=1}^{k} (1 + (k+2)^{q-1}) \eta_l^q f(t_l) \right]^{\frac{1}{q}} \times \\ & \times \Bigg[1 - (m-1) \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} (i+2)^{q-1} m \prod_{j=1}^{i-1} (1 + (j+2)^{q-1}) \eta_j^q f(t_j))^m F_1^q(s) f_p^m(s) ds - \\ & - (m-1)(k+3)^{(q-1)m} \times \prod_{j=1}^{k} (1 + (j+2)^{q-1} \eta_j^q f(t_j))^m \int_{t_k}^{t} F_1^q(s) f_p^m(s) ds \Bigg]^{\frac{1}{q(1-m)}} \end{aligned} \tag{2.3}$$

as long as the expression between the second brackets is positive, that is, on $(0, T_p)$, T_p *is the sup of all values of* t *for which*

$$\sum_{i=1}^{k} \int\limits_{t_{i-1}}^{t_i} (i+2)^{(q-1)m} \prod_{j=1}^{i-1} (1+(j+2)^{q-1})\eta_j^q f(t_j))^m F_1^q(s) f_p^m(s) ds -$$
$$-(m-1)(k+3)^{(q-1)m} \times \prod_{j=1}^{k} (1+(j+2)^{q-1}\eta_j^q f(t_j))^m \int\limits_{t_k}^{t} F_1^q(s) f_p^m(s) ds <$$
$$< \frac{1}{m-1};$$

(2) *If* $\frac{1}{2} < \alpha$, $-\frac{1}{2} < \gamma$, *then it holds that for* $t \in (t_k, t_{k+1}]$,

$$x(t) \leq \left[(k+3) f(t) \prod_{l=1}^{k} (1+(k+2))\eta_l^2 f(t_l) \right]^{\frac{1}{2}} \times$$
$$\times \left[1-(m-1) \sum_{i=1}^{k} \int\limits_{t_{i-1}}^{t_i} (i+2)m \prod_{j=1}^{i-1} (1+(j+2))\eta_j^2 f(t_j))^m F_1^2(s) f^m(s) ds - \right.$$
$$\left. -(m-1)(k+3)^m \times \prod_{j=1}^{k} (1+(j+2)\eta_j^2 f(t_j))^m \int\limits_{t_k}^{t} F_1^2(s) f^m(s) ds \right]^{\frac{1}{2(1-m)}}$$

as long as the expression between the second brackets is positive, that is, on $(0, T_2)$, T_2 *is the sup of all values of* t *for which*

$$\sum_{i=1}^{k} \int\limits_{t_{i-1}}^{t_i} (i+2)^m \prod_{j=1}^{i-1} (1+(j+2))\eta_j^2 f(t_j))^m F_1^2(s) f^m(s) ds -$$
$$-(m-1)(k+3)^m \times \prod_{j=1}^{k} (1+(j+2)\eta_j^2 f(t_j))^m \int\limits_{t_k}^{t} F_1^2(s) f^m(s) ds <$$
$$< \frac{1}{m-1},$$

where

$$f_p(t) = \sup\left\{a^q(t), C^{\frac{q}{p}}(p\alpha - p + 1, p\gamma)b^q(t)t^{q(\alpha+\gamma)-1}, d^q(t)\right\},$$
$$p \text{ and } q \text{ such that } \frac{1}{p} + \frac{1}{q} = 1,$$
$$f(t) \equiv f_2(t) = \sup\left\{a^2(t), C(2\alpha - 2 + 1, 2\gamma)b^2(t)t^{2(\alpha+\gamma)-1}, d^2(t)\right\},$$
$$\text{if } p = q = 2,$$
$$C(p\alpha - p + 1, p\gamma) = \frac{\Gamma(p\alpha - p + 1)\Gamma(p\gamma + 1)}{\Gamma(p\alpha - p + 1 + p\gamma + 1)}.$$

3. EXISTENCE OF MILD SOLUTIONS

In this section, we will derive the existence result concerning the PC-mild solution for the system (1.1) under some easily checked conditions.

We make the following assumptions.

[HA]: A is the infinitesimal generator of a compact C_0-semigroup $\{T(t), t \geq 0\}$ on X with domain $D(A)$.

[Hf]: (1) $f : J \times X \to X$ is strongly measurable with respect to t on J and for any x, $y \in X$ satisfying $\|x\|$, $\|y\| \leq \rho$ there exists a positive constant $L_f(\rho) > 0$ such that

$$\|f(t, x) - f(t, y)\| \leq L_f(\rho)\|x - y\|.$$

(2) There exists a positive constant $M_f > 0$ such that

$$\|f(t, x)\| \leq M_f(1 + \|x\|^m) \text{ for all } t \in J,\ x \in X,\ \text{some } m > 1.$$

[HI]: (1) The nonlinear map I_k: $X \to X$, $I_k(X)$ is a bounded subset of X, $k = 1, 2, \ldots, \delta$.

(2) There exist constants $h_k > 0$, such that

$$\|I_k(x) - I_k(y)\| \leq h_k\|x - y\|, \quad \text{for all} \quad x,\ y \in X,\ k = 1, 2, \ldots, \delta.$$

Theorem 3.1. *Under the assumptions* [HA], [Hf] *and* [HI], *system* (1.1) *has at least a PC-mild solution on J.*

Proof. Let $x_0 \in X$ be fixed. Define an operator H on $PC(J,X)$ which is given by

$$
\begin{aligned}
(Hx)(t) = T(t)x_0 &+ \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} T(t-s) s^n f(s, x(s))\, ds + \\
&+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} T(t-s) s^n f(s, x(s))\, ds + \\
&+ \sum_{0<t_k<t} T(t-t_k) I_k(x(t_k)).
\end{aligned} \tag{3.1}
$$

Using [HA] and [Hf], one can verify that H is a continuous mapping from $PC(J,X)$ to $PC(J,X)$ for $x \in PC(J,X)$. In fact, for $0 \le \tau < t \le t_1$, it comes from [HA] and the following inequality

$$
\begin{aligned}
\|(Hx)(t) - (Hx)(\tau)\| &\le \|T(t)x_0 - T(\tau)x_0\| + \\
&+ \frac{1}{\Gamma(\alpha)} \int_{\tau}^{t} \|(t-s)^{\alpha-1} T(t-s) s^n f(s, x(s))\| ds + \\
&+ \frac{1}{\Gamma(\alpha)} \int_{0}^{\tau} \|(t-s)^{\alpha-1} [T(t-s) - T(\tau - s)] s^n f(s, x(s))\| ds \le \\
&\le M\|T(t-\tau)x_0 - x_0\| + \\
&+ \frac{t^n M \|f\|_{PC}}{\Gamma(\alpha)} \int_{\tau}^{t} (t-s)^{\alpha-1} ds + \\
&+ \frac{\tau^n M \|T(t-\tau) - I\| \|f\|_{PC}}{\Gamma(\alpha)} \int_{0}^{\tau} (t-s)^{\alpha-1} ds \le \\
&\le M\|T(t-\tau)x_0 - x_0\| + \\
&+ \frac{t_1^n M \|f\|_{PC}}{\Gamma(\alpha+1)} (t-\tau)^{\alpha} + \\
&+ \left[\frac{t_1^n M \|f\|_{PC}}{\Gamma(\alpha+1)} |t^{\alpha} - (t-\tau)^{\alpha}| \right] \|T(t-\tau) - I\|
\end{aligned} \tag{3.2}
$$

that $Hx \in C([0, t_1], X)$.

With analogous arguments we can obtain $Hx \in C([t_k, t_{k+1}], X)$, $k = 0, 1, 2, \ldots, \delta$. That is $Hx \in PC(J,X)$.

(1) H is a continuous operator on $PC(J,X)$.

Let $x_1, x_2 \in PC(J,X)$ and $\|x_1 - x_2\|_{PC} \le 1$, then $\|x_2\|_{PC} \le 1 + \|x_1\|_{PC} = \rho$. By assumptions [HA], [Hf] and [HI], we obtain

$$\begin{aligned}
&\|(Hx_1)(t) - (Hx_2)(t)\| \le \\
&\le \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} \|(t_k - s)^{\alpha-1} T(t-s) s^n [f(s, x_1(s)) - f(s, x_2(s))]\| ds + \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} \|(t-s)^{\alpha-1} T(t-s) s^n [f(s, x_1(s)) - f(s, x_2(s))]\| ds + \\
&\quad + \sum_{0<t_k<t} \|T(t-t_k)[I_k(x_1(t_k)) - I_k(x_2(t_k))]\| \le \\
&\le \frac{ML_f(\rho)}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} s^n \|x_1(s) - x_2(s)\| ds + \\
&\quad + \frac{ML_f(\rho)}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} s^n \|x_1(s) - x_2(s)\| ds + M \sum_{0<t_k<t} h_k \|x_1(t_k) - x_2(t_k)\| \le \\
&\le \left(\int_0^t (t-s)^{\alpha-1} s^n ds \right) \left[\frac{ML_f(\rho)}{\Gamma(\alpha)} + M \sum_{0<t_k<t} h_k \right] \|x_1 - x_2\|_{PC}.
\end{aligned} \tag{3.3}$$

Using Lemma 2.6, one can deduce that

$$\begin{aligned}
\|Hx_1 - Hx_2\|_{PC} &\le \frac{\Gamma(\alpha) \cdot \Gamma(n+1)}{\Gamma(\alpha+n+1)} t^{\alpha+n} \left[\frac{ML_f(\rho)}{\Gamma(\alpha)} + M \sum_{k=1}^{\delta} h_k \right] \|x_1 - x_2\|_{PC} \le \\
&\le L \|x_1 - x_2\|_{PC},
\end{aligned}$$

where

$$L = M b^{\alpha+n} \frac{\Gamma(\alpha) \cdot \Gamma(n+1)}{\Gamma(\alpha+n+1)} \left[\frac{L_f(\rho)}{\Gamma(\alpha)} + \sum_{k=1}^{\delta} h_k \right].$$

(2) H is a compact operator on $PC(J,X)$.

Let $\mathfrak{B}$ be a bounded subset of $PC(J,X)$, there exists a constant $\mu > 0$ such that $\|x\|_{PC} \le \mu$ for all $x \in \mathfrak{B}$. Using [HI], there exists a constant N such that $\|I_k(x(t))\| \le N$ for all $x \in \mathfrak{B}$, $t \in J$, $k = 1, 2, \ldots, \delta$. Also using [Hf], there exists a constant ω such that $\|f(t, x(t)\| \le M_f(1 + \|x\|_{PC}^m) \le M_f(1 + \mu^m) \equiv \omega$ for all $x \in \mathfrak{B}$,

$t \in J$. Further, $H\mathfrak{B}$ is a bounded subset of $PC(J,X)$. In fact, let $x \in \mathfrak{B}$, we have

$$\begin{aligned}\|(Hx)(t)\| &\le M\|x_0\| + \frac{M\omega}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} s^n ds + \\ &+ \frac{M\omega}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} s^n ds + M \sum_{0<t_k<t} N \le \\ &\le M\|x_0\| + MN\delta + \frac{M\omega}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} s^n ds \le \\ &\le M\|x_0\| + MN\delta + \frac{\omega M \Gamma(n+1)}{\Gamma(\alpha+n+1)} b^{\alpha+n}.\end{aligned}$$

Hence $H\mathfrak{B}$ is bounded.

Define

$$\Pi = H\mathfrak{B} \quad \text{and} \quad \Pi(t) = \{(Hx)(t) \mid x \in \mathfrak{B})\} \text{ for } t \in J.$$

Clearly, $\Pi(0) = \{x_0\}$ is compact, hence, it is only necessary to check that $\Pi(t) = \{(Hx)(t) \mid x \in \mathfrak{B}\}$ for $t \in (0,b]$ is also compact. For $0 < \varepsilon < t \le b$, define

$$\Pi_\varepsilon(t) \equiv (H_\varepsilon \mathfrak{B})(t) = \{(H_\varepsilon x)(t) \mid x \in \mathfrak{B}\} \tag{3.4}$$

and the operator H_ε is defined by

$$\begin{aligned}(H_\varepsilon x)(t) &= T(\varepsilon)T(t-\varepsilon)x_0 + \\ &+ T(\varepsilon)\frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} T(t-\varepsilon-s) s^n f(s, x(s))\, ds + \\ &+ T(\varepsilon)\frac{1}{\Gamma(\alpha)} \int_{t_k}^{t-\varepsilon} (t-s)^{\alpha-1} T(t-\varepsilon-s) s^n f(s, x(s))\, ds + \\ &+ T(\varepsilon) \sum_{0<t_k<t} T(t-\varepsilon-t_k) I_k(x(t_k)) = \\ &= T(t)x_0 + \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} T(t-s) s^n f(s, x(s))\, ds + \\ &+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t-\varepsilon} (t-s)^{\alpha-1} T(t-s) s^n f(s, x(s))\, ds + \\ &+ \sum_{0<t_k<t} T(t-t_k) I_k(x(t_k)),\end{aligned} \tag{3.5}$$

from which implies that $\Pi_\varepsilon(t)$ is relatively compact for $t \in (\varepsilon, b]$ due to $\{T(t), t \geq 0\}$ is a compact semigroup.

For interval $(0, t_1]$, (3.4) reduces to

$$\Pi_\varepsilon(t) \equiv (H_\varepsilon \mathfrak{B})(t) = \{(H_\varepsilon x)(t) \mid x \in \mathfrak{B}\}.$$

Combine with (3.1) and (3.5), we can deduce

$$\begin{aligned}\sup_{x\in\mathfrak{B}} \|(Hx)(t) - (H_\varepsilon x)(t)\| &\leq \frac{1}{\Gamma(\alpha)} \left\| \int_{t-\varepsilon}^{t} (t-s)^{\alpha-1} T(t-s) s^n f(s, x(s))\, ds \right\| \leq \\ &\leq \frac{t^n \omega M}{\Gamma(\alpha)} \int_{t-\varepsilon}^{t} (t-s)^{\alpha-1} ds \leq \\ &\leq \frac{b^n \omega M \varepsilon^\alpha}{\Gamma(\alpha+1)}.\end{aligned}$$

It shows that the set $\Pi(t)$ can be approximated to an arbitrary degree of accuracy by a relatively compact set for $t \in (0, t_1]$. Hence, $\Pi(t)$ itself is a relatively compact set for $t \in (0, t_1]$.

For interval $(t_1, t_2]$, define

$$\begin{aligned}\Pi(t_1+0) &\equiv \Pi(t_1-0) + I_1(\Pi(t_1-0)) = \Pi(t_1) + I_1(\Pi(t_1)) = \\ &= \{(Hx)(t_1) + I_1(x(t_1)) \mid x \in \mathfrak{B}\}.\end{aligned}$$

By assumption [HI], one can verify that $I_1(\Pi(t_1))$ is relatively compact. Hence, $\Pi(t_1+0)$ is relatively compact. Then (3.4) reduces to

$$\begin{aligned}&\Pi_\varepsilon(t) \equiv (H_\varepsilon \mathfrak{B})(t) = \\ &= \left\{ (Hx)(t_1+0) + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t-\varepsilon} (t-s)^{\alpha-1} T(t-s) s^n f(s, x(s))\, ds \mid x \in \mathfrak{B} \right\}.\end{aligned}$$

By elementary computation again, we have

$$\sup_{x\in\mathfrak{B}} \|(Hx)(t) - (H_\varepsilon x)(t)\| \leq \frac{\omega M}{\Gamma(\alpha)} \int_{t-\varepsilon}^{t} (t-s)^{\alpha-1} s^n ds \leq \frac{b^n \omega M \varepsilon^\alpha}{\Gamma(\alpha+1)}.$$

Hence, $\Pi(t)$ itself is relatively compact set for $t \in (t_1, t_2]$.

In general, for any given t_k, $k = 1, 2, \ldots, \delta$, we define that $x(t_i + 0) = x_i$, and

$$\begin{aligned}\Pi(t_k+0) &\equiv \Pi(t_k-0) + I_k(\Pi(t_k-0)) = \\ &= \Pi(t_k) + I_k(\Pi(t_k)) = \\ &= \{(Hx)(t_k) + I_k(x(t_k)) \mid x \in \mathfrak{B}\}, \quad k = 1, 2, \ldots, \delta.\end{aligned}$$

By assumption [HI] again, $I_k(\Pi(t_k)$ is relatively compact and the associated $\Pi_\varepsilon(t)$ over the interval $(t_k, t_{k+1}]$ is given by

$$\Pi_\varepsilon(t) \equiv (H_\varepsilon \mathfrak{B})(t) =$$
$$= \left\{ (Hx)(t_k+0) + \frac{1}{\Gamma(\alpha)} \int\limits_{t_k}^{t-\varepsilon} (t-s)^{\alpha-1} T(t-s) s^n f(s, x(s))\, ds \mid x \in \mathfrak{B} \right\}.$$

Thus, we have

$$\sup_{x\in\mathfrak{B}} \|(Hx)(t) - (H_\varepsilon x)(t)\| \leq \frac{b^n \omega M \varepsilon^\alpha}{\Gamma(\alpha+1)}.$$

Hence, $\Pi(t)$ itself is a relatively compact set for $t \in (t_k, t_{k+1}]$.

Now, we repeat the procedures till the time interval which is expanded. Thus, we can obtain that the set $\Pi(t)$ itself is relatively compact for $t \in J \setminus \{t_1, t_2, \ldots, t_\delta\}$ and $\Pi(t_k+0)$ is relatively compact for t_k, $k = 1, 2, \ldots, \delta$.

(3) Π is equicontinuous on the interval (t_k, t_{k+1}), $k = 1, 2, \ldots, \delta$.

For interval $(0, t_1)$, we note that for $t_1 > h > 0$,

$$\|(Hx)(h) - (Hx)(0)\| \leq \|T(h) - I\|\|x_0\| + \omega M \frac{\Gamma(n+1)}{\Gamma(\alpha+n+1)} h^{\alpha+n},$$

and for $t_1 \geq t+h \geq t \geq \gamma \geq 0$, $\gamma < h$ and $x \in \mathfrak{B}$,

$$(Hx)(t+h) - (Hx)(t) = (T(t+h) - T(t))x_0 +$$
$$+ \frac{1}{\Gamma(\alpha)} \int\limits_{t}^{t+h} (t+h-s)^{\alpha-1} T(t+h-s) s^n f(s, x(s))\, ds +$$
$$+ \frac{1}{\Gamma(\alpha)} \int\limits_{t-\gamma}^{t} (t+h-s)^{\alpha-1} [T(t+h-s) - T(t-s)] s^n f(s, x(s))\, ds +$$
$$+ \frac{1}{\Gamma(\alpha)} \int\limits_{t-\gamma}^{t} [(t+h-s)^{\alpha-1} - (t-s)^{\alpha-1}] T(t-s) s^n f(s, x(s))\, ds +$$
$$+ \frac{1}{\Gamma(\alpha)} \int\limits_{0}^{t-\gamma} (t+h-s)^{\alpha-1} [T(t+h-s) - T(t-s)] s^n f(s, x(s))\, ds +$$
$$+ \frac{1}{\Gamma(\alpha)} \int\limits_{0}^{t-\gamma} [(t+h-s)^{\alpha-1} - (t-s)^{\alpha-1}] T(t-s) s^n f(s, x(s))\, ds,$$

hence,

$$\begin{aligned}\|(Hx)(t+h)-(Hx)(t)\| \leq M\|T(h)-I\|\|x_0\|+ \\ &+\frac{\omega M t_1^n}{\Gamma(\alpha+1)}h^\alpha+ \\ &+\omega M t_1^n\frac{|h^\alpha-(h+\gamma)^\alpha|}{\Gamma(\alpha+1)}\|T(h)-I\|+ \\ &+\omega M t_1^n\frac{1}{\Gamma(\alpha)}\int\limits_{t-\gamma}^{t}|(t+h-s)^{\alpha-1}-(t-s)^{\alpha-1}|ds+ \\ &+\omega M t_1^n\frac{|(t+h)^\alpha-(\gamma+h)^\alpha|}{\Gamma(\alpha+1)}\|T(h)-I\|+ \\ &+\omega M t_1^n\frac{1}{\Gamma(\alpha)}\int\limits_{0}^{t-\gamma}|(t+h-s)^{\alpha-1}-(t-s)^{\alpha-1}|ds.\end{aligned} \tag{3.6}$$

Since $\|T(h)-I\|\to 0$, $|(t+h-s)^{\alpha-1}-(t-s)^{\alpha-1}|\to 0$ as $h\to 0$, thus the right hand side of (3.6) can be made as small as desired by choosing h sufficiently small. Hence, $\Pi(t)$ is equicontinuous in interval $(0,t_1)$.

In general, for time interval (t_k,t_{k+1}), $k=1,2,\ldots,\delta$, we similarly obtain the following inequalities

$$\begin{aligned}\|(Hx)(t+h)-(Hx)(t)\| \leq M\|T(h)-I\|\|x_k\|+ \\ &+\frac{\omega M t_{k+1}^n}{\Gamma(\alpha+1)}h^\alpha+ \\ &+\omega M t_{k+1}^n\frac{|h^\alpha-(h+\gamma)^\alpha|}{\Gamma(\alpha+1)}\|T(h)-I\|+ \\ &+\omega M t_{k+1}^n\frac{1}{\Gamma(\alpha)}\int\limits_{t-\gamma}^{t}|(t+h-s)^{\alpha-1}-(t-s)^{\alpha-1}|ds+ \\ &+\omega M t_{k+1}^n\frac{|(t+h)^\alpha-(\gamma+h)^\alpha|}{\Gamma(\alpha+1)}\|T(h)-I\|+ \\ &+\omega M t_{k+1}^n\frac{1}{\Gamma(\alpha)}\int\limits_{t_k}^{t-\gamma}|(t+h-s)^{\alpha-1}-(t-s)^{\alpha-1}|ds.\end{aligned}$$

With analogous arguments, one can verify that Π is also equicontinuous on the interval (t_k,t_{k+1}), $k=1,2,\ldots,\delta$.

Now, we repeat the procedures till the time interval which is expanded. Thus we obtain that the set $\Pi(t)$ itself is relatively compact for $t\in J\backslash\{t_1,\ldots,t_\delta\}$ and $\Pi(t_k+0)$ is relatively compact for $t_k\in\{t_1,\ldots,t_\delta\}$.

(4) H has a fixed point in $PC(J,X)$.

According to Leray-Schauder fixed point theorem, it suffices to show the following set

$$\big\{x \in PC(J,X) \mid x = \sigma Hx, \sigma \in [0,1]\big\}$$

is a bounded subset of $PC(J,X)$. In fact, let $x \in \{x \in PC(J,X) \mid x = H(\sigma x), \sigma \in [0,1]\}$, we have

$$\begin{aligned}
\|x(t)\| = \|H(\sigma x(t))\| \le \\
&\le \|T(t)(\sigma x_0)\| + \\
&+ \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int\limits_{t_{k-1}}^{t_k} \|(t_k - s)^{\alpha-1} T(t-s) s^n f\left(s, \sigma x(s)\right)\| ds + \\
&+ \frac{1}{\Gamma(\alpha)} \int\limits_{t_k}^{t} \|(t-s)^{\alpha-1} T(t-s) s^n f\left(s, \sigma x(s)\right)\| ds + \\
&+ \sum_{0<t_k<t} \|T(t-t_k) I_k(\sigma x(t_i))\| \le \\
&\le \sigma M \|x_0\| + M M_f \frac{1}{\Gamma(\alpha)} \int\limits_0^t (t-s)^{\alpha-1} s^n (1 + \sigma \|x(s)\|^m) ds + \\
&+ M \sum_{0<t_k<t} (\|I_k(0)\| + \sigma h_k \|x(t_k)\|) \le \\
&\le M\Bigg(\|x_0\| + M_f \frac{\Gamma(n+1)}{\Gamma(\alpha+n+1)} t^{\alpha+n} + \sum_{k=1}^{\delta} \|I_i(0)\|\Bigg) + \\
&+ \frac{M M_f}{\Gamma(\alpha)} \int\limits_0^t (t-s)^{\alpha-1} s^{\gamma} s^{n-\gamma} \|x(s)\|^m ds + \\
&+ M \sum_{0<t_k<t} h_k \|x(t_k)\| \le \\
&\le M\Bigg(\|x_0\| + M_f \frac{\Gamma(n+1)}{\Gamma(\alpha+n+1)} t^{\alpha+n} + \sum_{k=1}^{\delta} \|I_k(0)\|\Bigg) + \\
&+ t^{n-\gamma} \frac{M M_f}{\Gamma(\alpha)} \int\limits_0^t (t-s)^{\alpha-1} s^{\gamma} \|x(s)\|^m ds + \\
&+ M \sum_{0<t_k<t} h_k \|x(t_k)\|.
\end{aligned}$$

Denote

$$f_p(t) = \sup\Bigg\{M^q\Bigg(\|x_0\| + M_f\frac{\Gamma(n+1)}{\Gamma(\alpha+n+1)}t^{\alpha+n} + \sum_{k=1}^{\delta}\|I_k(0)\|\Bigg)^q,$$

$$C^{\frac{q}{p}}(p\alpha-p+1,p\gamma)t^{(n-\gamma)q}\Bigg(\frac{MM_f}{\Gamma(\alpha)}\Bigg)^q t^{q(\alpha+\gamma)-1}, M^q\Bigg\},$$

$$p \text{ and } q \text{ such that } \frac{1}{p}+\frac{1}{q}=1,$$

$$f(t) \equiv f_2(t) = \sup\Bigg\{M^2\Bigg(\|x_0\| + M_f\frac{\Gamma(n+1)}{\Gamma(\alpha+n+1)}t^{\alpha+n} + \sum_{k=1}^{\delta}\|I_k(0)\|\Bigg)^2,$$

$$C(2\alpha-2+1,2\gamma)t^{(n-\gamma)2}\Bigg(\frac{MM_f}{\Gamma(\alpha)}\Bigg)^2 t^{2(\alpha+\gamma)-1}, M^2\Bigg\}.$$

(i) If $\frac{1}{2} \geq \alpha > 0$, $-\frac{1}{2} \geq \gamma > -1$, by (1) of Lemma 2.7, it holds that for each $(t_k, t_{k+1}]$,

$$\|x(t)\| \leq \sup_{t\in[0,T_p]}\Bigg\{\Bigg[(k+3)^{q-1}f_p(t)\prod_{l=1}^{k}(1+(k+2)^{q-1})h_l^q f(t_l)\Bigg]^{\frac{1}{q}}\Bigg\} \equiv M_k^{1*},$$

where T_p is the sup of all values of t for which

$$\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}(i+2)^{(q-1)m}\prod_{j=1}^{i-1}(1+(j+2)^{q-1})h_j^q f(t_j))^m f_p^m(s)ds-$$

$$-(m-1)(k+3)^{(q-1)m}\times\prod_{j=1}^{k}(1+(j+2)^{q-1}h_j^q f(t_j))^m\int_{t_k}^{t} f_p^m(s)ds < \frac{1}{m-1}.$$

(ii) If $\frac{1}{2} < \alpha < 1$, $-\frac{1}{2} < \gamma$, by (2) of Lemma 2.7, it holds that for each $(t_k, t_{k+1}]$,

$$\|x(t)\| \leq \sup_{t\in[0,T_2]}\Bigg\{\Bigg[(k+3)f(t)\prod_{l=1}^{k}(1+(k+2))h_l^2 f(t_l)\Bigg]^{\frac{1}{2}}\Bigg\} \equiv M_k^{2*},$$

where T_2 is the sup of all values of t for which

$$\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}((i+2)^m)\prod_{j=1}^{i-1}(1+(j+2))h_j^2 f(t_j))^m f^m(s)ds-$$

$$-(m-1)(k+3)^m\times\prod_{j=1}^{k}(1+(j+2)h_j^2 f(t_j))^m\int_{t_k}^{t} f^m(s)ds < \frac{1}{m-1}.$$

Set $M^{1*} = \max\{M_1^{1*}, M_2^{1*}, \ldots, M_\delta^{1*}\}$, $M^{2*} = \max\{M_1^{2*}, M_2^{2*}, \ldots, M_\delta^{2*}\}$. We denote $M^* = \max\{M^{1*}, M^{2*}\}$. Then we have

$$\|x\|_{PC} \leq M^* \text{ for all } x \in \{x \in PC(J, X) \mid x = \sigma H x, \sigma \in [0, 1]\}.$$

Thus, $\{x \in PC(J, X) \mid x = \sigma H x, \sigma \in [0, 1]\}$ is a bounded subset of $PC(J, X)$. By the Leray-Schauder fixed point theorem, we obtain that H has a fixed point in $PC(J, X)$. This completes the system (1.1) has at least a PC-mild solution on J. □

At last, an example is given to illustrate our theory. Consider the following impulsive fractional differential equations

$$\begin{cases} D_t^{\frac{1}{3}} x(t, y) = \frac{\partial^2}{\partial y^2} x(t, y) + t x^2(t, y) + t \sin(t, y), \ t \in (0, 1] \setminus \{\frac{1}{2}\}, \\ \Delta x(t_1, y) = -x(t_1, y), \ t_1 = \frac{1}{2}, \ y \in \Omega = (0, \pi), \\ x(t, y) \mid_{y \in \partial\Omega} = 0, \ t > 0, \ x(0, y) = 0, \ y \in \Omega. \end{cases} \tag{3.7}$$

Let $X = L^2([0, \pi])$. Define

$$D(A) = \left\{ x \in X \mid \frac{\partial x}{\partial y}, \frac{\partial^2 x}{\partial y^2} \text{ and } x(0) = x(\pi) = 0 \right\} \text{ and } Ax = -\frac{\partial^2}{\partial y^2} x \text{ for } x \in D(A)$$

which can determine a compact C_0-semigroup $\{T(t), t \geq 0\}$ in $L^2([0, \pi])$ such that $\|T(t)\| \leq 1$.

Denote $x(\cdot)(y) = x(\cdot, y)$, $\sin(\cdot)(y) = \sin(\cdot, y)$, $f(\cdot, x(\cdot))(y) = x^2(\cdot, y) + \sin(\cdot, y)$, $I_1(x(t_1))(y) = -x(t_1, y)$. Thus, problem (3.7) can be rewritten as

$$\begin{cases} D_t^\alpha x(t) = Ax(t) + t^n f(t, x(t)), \ \alpha = \frac{1}{3} \in (0, \frac{1}{2}], n = 1, t \in (0, 1] \setminus \{t_k\}, \\ \Delta x(t_k) = I_k(x(t_k)), \ t_k = \frac{1}{2}, \ k = 1, \\ x(0) = 0. \end{cases} \tag{3.8}$$

Obviously, all the assumptions in Theorem 3.1 are satisfied. Our results can be used to solve problem (3.7).

Acknowledgments

JinRong Wang acknowledge support from the National Natural Science Foundation of Guizhou Province (2010, No. 2142) and the Introducing Talents Foundation for the Doctor of Guizhou University (2009, No. 031).

W. Wei acknowledge support from the National Natural Science Foundation of China (No. 10961009).

YanLong Yang acknowledge support from the Youth Teachers Natural Science Foundation of Guizhou University (2009, No. 083).

REFERENCES

[1] N.U. Ahmed, *Optimal impulsive control for impulsive systems in Banach space*, Int. J. Differ. Equ. Appl. **1** (2000), 37–52.

[2] N.U. Ahmed, *Some remarks on the dynamics of impulsive systems in Banach space*, Mathematical Anal. **8** (2001), 261–274.

[3] N.U Ahmed, K.L. Teo, S.H. Hou, *Nonlinear impulsive systems on infinite dimensional spaces*, Nonlinear Anal. **54** (2003), 907–925.

[4] N.U. Ahmed, *Existence of optimal controls for a general class of impulsive systems on Banach space*, SIAM J. Control Optim. **42** (2003), 669–685.

[5] N.U. Ahmed, *Optimal feedback control for impulsive systems on the space of finitely additive measures*, Publ. Math. Debrecen **70** (2007), 371–393.

[6] R.P. Agarwal, M. Benchohra, B.A. Slimani, *Existence results for differential equations with fractional order and impulses*, Mem. Differential Equations Math. Phys. **44** (2008), 1–21.

[7] B. Ahmad, S. Sivasundaram, *Existence results for nonlinear impulsive hybrid boundary value problems involving fractional differential equations*, Nonlinear Anal. **3** (2009), 251–258.

[8] M. Benchohra, J. Henderson, S.K. Ntouyas, A. Ouahab, *Multiple solutions for impulsive semilinear functional and neutral functional differential equations in Hilbert space*, J. Inequal. Appl. **2005** (2005), 189–205.

[9] M. Benchohra, J. Henderson, S.K. Ntouyas, *Impulsive Differential Equations and Inclusions*, vol. 2, Hindawi Publishing Corporation, New York, 2006.

[10] M. Benchohra, B.A. Slimani, *Existence and uniqueness of solutions to impulsive fractional differential equations*, Electron. J. Differential Equations **10** (2009), 1–11.

[11] M. Benchohra, F. Berhoun, *Impulsive fractional differential equations with variable times*, Computers and Mathematics with Applications, in press. doi:10.1016/j.camwa.2009.05.016.

[12] M. Benchohra, D. Seba, *Impulsive fractional differential equations in Banach spaces*, Electron. J. Qual. Theory Differ. Equ. Spec. Ed. I **8** (2009), 1–14.

[13] K. Balachandran, S. Kiruthika, *Existence of solutions of abstract fractional impulsive semilinear evolution equations*, Electron. J. Qual. Theory Differ. Equ. **4** (2010), 1–12.

[14] D.D. Bainov, P.S. Simeonov, *Impulsive Differential Equations: Periodic Solutions and Applications*, Longman Scientific and Technical Group. Limited, New York, 1993.

[15] Y.-K. Chang, J.J. Nieto, *Existence of solutions for impulsive neutral integro-differential inclusions with nonlocal initial conditions via fractional operators*, Numer. Funct. Anal. Optim. **30** (2009), 227–244.

[16] Diethelm, A.D. Freed, *On the Solution of Nonlinear Fractional Order Differential Equations Used in the Modeling of Viscoelasticity*, [in:] F. Keil, W. Mackens, H. Voss, J. Werther (Eds.), Scientific Computing in Chemical Engineering II-Computational Fluid Dynamics, Reaction Engineering and Molecular Properties, Springer-Verlag, Heidelberg, 1999, 217–224.

[17] M.M. El-Borai, *Semigroup and some nonlinear fractional differential equations*, Appl. Math. Comput. **149** (2004), 823–831.

[18] M.M. El-Borai, *Evolution equations without semigroup*, Appl. Math. Comput. **149** (2004), 815–821.

[19] L. Gaul, P. Klein, S. Kempfle, *Damping description involving fractional operators*, Mech. Syst. Signal Process. **5** (1991), 81–88.

[20] W.G. Glockle, T.F. Nonnenmacher, *A fractional calculus approach of self-similar protein dynamics*, Biophys. J. **68** (1995), 46–53.

[21] R. Hilfer, *Applications of Fractional Calculus in Physics*, World Scientific, Singapore, 2000.

[22] L. Hu, Y. Ren, R. Sakthivel, *Existence and uniqueness of mild solutions for semilinear integro-differential equations of fractional order with nonlocal initial conditions and delays*, Semigroup Forum **79** (2009), 507–514.

[23] S. Hu, N.S. Papageorgiou, *Handbook of Multivalued Analysis (Theory)*, Kluwer Academic Publishers, Dordrecht, Boston, London, 1997.

[24] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*, [in:] North-Holland Mathematics Studies, vol. 204, Elsevier Science B.V, Amsterdam, 2006.

[25] V. Lakshmikantham, D.D. Bainov, P.S. Simeonov, *Theory of impulsive differential equations*, World Scientific, Singapore-London, 1989.

[26] V. Lakshmikantham, *Theory of fractional differential equations*, Nonlinear Anal. **60** (2008), 3337–3343.

[27] V. Lakshmikantham, A.S. Vatsala, *Basic theory of fractional differential equations*, Nonlinear Anal. **69** (2008), 2677–2682.

[28] V. Lakshmikantham, S. Leela, J. Vasundhara Devi, *Theory of Fractional Dynamic Systems*, Cambridge Scientific Publishers, 2009.

[29] J. Liu, *Nonlinear impulsive evolution equations*, Dyn. Contin. Discrete Impuls. Syst. **6** (1999), 77–85.

[30] J. Liang, J.H. Liu, T.J. Xiao, *Nonlocal impulsive problems for nonlinear differential equations in Banch spaces*, Math. Comput. Modelling **49** (2009), 798–804.

[31] K.S. Miller, B. Ross, *An Introduction to the Fractional Calculus and Differential Equations*, John Wiley, New York, 1993.

[32] X. Li, J. Yong, *Optimal control theory for infinite dimensional systems*, Birkhauser Boston, 1995.

[33] F. Mainardi, *Fractional Calculus, Some Basic Problems in Continuum and Statistical Mechanics*, [in:] A. Carpinteri, F. Mainardi (Eds.), Fractals and Fractional Calculus in Continuum Mechanics, Springer-Verlag, Wien, 1997, 291–348.

[34] G.M. Mophou, G.M. N'Guérékata, *Mild solutions for semilinear fractional differential equations*, Electron. J. Differ. Equ. **21** (2009), 1–9.

[35] G.M. Mophou, G.M. N'Guérékata, *Existence of mild solution for some fractional differential equations with nonlocal conditions*, Semigroup Forum **79** (2009), 315–322.

[36] G.M. Mophou, *Existence and uniqueness of mild solution to impulsive fractional differetial equations*, Nonlinear Anal. **72** (2010), 1604–1615.

[37] F. Metzler, W. Schick, H.G. Kilian, T.F. Nonnenmache, *Relaxation in filled polymers: A fractional calculus approach*, J. Chem. Phys. **103** (1995), 7180–7186.

[38] J.J. Nieto, R. Rodriguez-Lopez, *Boundary value problems for a class of impulsive functional equations*, Comput. Math. Appl. **55** (2008), 2715–2731.

[39] J.J. Nieto, D. O'Regan, *Variational approach to impulsive differential equations*, Nonlinear Anal. **10** (2009), 680–690.

[40] N. Özdemir, D. Karadeniz, B.B. İskender, *Fractional optimal control problem of a distributed system in cylindrical coordinates*, Phys. Lett. A **373** (2009), 221–226.

[41] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, 1999.

[42] N.-E. Tatar, *An impulsive nonlinear singular version of the Gronwall-Bihari inequality*, J. Inequal. Appl. **2006** (2006), Article ID 84561, 1–12.

[43] J.R. Wang, X. Xiang, W. Wei, *Linear impulsive periodic system with time-varying generating operators on Banach space*, Adv. Difference Equ. **2007** (2007), Article ID 26196, 1–16.

[44] J.R. Wang, X. Xiang, W. Wei, Q. Chen, *Existence and global asymptotical stability of periodic solution for the T-periodic logistic system with time-varying generating operators and T_0-periodic impulsive perturbations on Banach spaces*, Discrete Dyn. Nat. Soc. **2008** (2008), Article ID 524945, 1–16.

[45] J.R. Wang, X. Xiang, Y. Peng, *Periodic solutions of semilinear impulsive periodic system on Banach space*, Nonlinear Anal. **71** (2009), e1344–e1353.

[46] J.R. Wang, X. Xiang, W. Wei, *Periodic solutions of a class of integrodifferential impulsive periodic systems with time-varying generating operators on Banach space*, Electron. J. Qual. Theory Differ. Equ. **4** (2009), 1–17.

[47] J.R. Wang, X. Xiang, W. Wei, *A class of nonlinear integrodifferential impulsive periodic systems of mixed type and optimal controls on Banach space*, J. Appl. Math. Comput. (2009). doi:10.1007/s12190-009-0332-8.

[48] J.R. Wang, W. Wei, Y.L. Yang, *Fractional nonlocal integrodifferential equations of mixed type with time-varying generating operators and optimal control*, Opuscula Math. **30** (2010) 2, 217–234.

[49] J.R. Wang, Y.L. Yang, W. Wei, *Nonlocal impulsive problems for fractional differential equations with time-varying generating operators in Banach spaces*, Opuscula Math. **30** (2010) 3, 361–381.

[50] W. Wei, X. Xiang, Y. Peng, *Nonlinear impulsive integro-differential equation of mixed type and optimal controls*, Optimization **55** (2006), 141–156.

[51] W. Wei, X. Xiang, *Optimal feedback control for a class of nonlinear impulsive evolution equations*, Chinese Journal of Engineering Mathematics **23** (2006), 333–342.

[52] Y. Peng, X. Xiang, W. Wei, *Optimal feedback control for a class of strongly nonlinear impulsive evolution equations*, Comput. Math. Appl. **52** (2006), 759–768.

[53] X. Yu, X. Xiang, W. Wei, *Solution bundle for class of impulsive differential inclusions on Banach spaces*, J. Math. Anal. Appl. **327** (2007), 220–232.

[54] T. Yang, *Impulsive Control Theory*, Springer-Verlag, Berlin, Heidelberg, 2001.

[55] E. Zeidler, *Nonlinear Functional Analysis and its Application II/A*, Springer-Verlag, New York, 1990.

JinRong Wang
wjr9668@126.com

Guizhou University
College of Science
Guiyang, Guizhou 550025, P.R. China

YanLong Yang
yylong@163.com

Guizhou University
College of Technology
Guiyang, Guizhou 55004, P.R. China

W. Wei
wwei@gzu.edu.cn

Guizhou University
College of Science
Guiyang, Guizhou 550025, P.R. China

Received: March 25, 2010.
Revised: April 12, 2010.
Accepted: April 17, 2010.