A NOTE ON GLOBAL ALLIANCES IN TREES

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Abstract. For a graph $G = (V, E)$, a set $S \subseteq V$ is a dominating set if every vertex in $V - S$ has at least a neighbor in S. A dominating set S is a global offensive (respectively, defensive) alliance if for each vertex in $V - S$ (respectively, in S) at least half the vertices from the closed neighborhood of v are in S. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set of G, and the global offensive alliance number $\gamma_o(G)$ (respectively, global defensive alliance number $\gamma_a(G)$) is the minimum cardinality of a global offensive alliance (respectively, global deffensive alliance) of G. We show that if T is a tree of order n, then $\gamma_o(T) \leq 2\gamma(T) - 1$ and if $n \geq 3$, then $\gamma_o(T) \leq \frac{3}{2}\gamma_a(T) - 1$. Moreover, all extremal trees attaining the first bound are characterized.

Keywords: global defensive alliance, global offensive alliance, domination, trees.

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1. INTRODUCTION

Let $G = (V, E)$ be a finite and simple graph of order n. The *open neighborhood* of a vertex $v \in V$ is $N(v) = \{u \in V \mid uv \in E\}$ and the *closed neighborhood* of v is $N[v] = N(v) \cup \{v\}$. If $S \subset V$, then $N(S) = \cup_{x \in S} N(x)$, $N[S] = \cup_{x \in S} N[x]$ and the subgraph induced by S in G is denoted $G[S]$. The *degree* of v, denoted by $\deg_G(v)$, is the size of its open neighborhood. A vertex of degree one is called a *pendent vertex* or a *leaf* and its neighbor is called a *support* vertex. If v is a support vertex, then L_v will denote the set of the leaves attached at v. We also denote the set of leaves of a graph G by $L(G)$, the set of support vertices by $S(G)$, and let $|L(G)| = \ell$, $|S(G)| = s$. A *double star* $S_{p,q}$ is a tree T containing exactly two vertices that are not leaves. Denote by T_x the subtree induced by a vertex x and its descendants in a rooted tree T.

For a graph $G = (V, E)$, a set of vertices S is a *dominating set* if every vertex in $V - S$ has at least a neighbor in S. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of G. For terminology not defined here and a thorough treatment of domination and its variations, see the books [3, 4].

In [5] Hedetniemi, Hedetniemi, and Kristiansen introduced several types of alliances in graphs, including the defensive and offensive alliances we consider here. A dominating set S of G is called a *global defensive alliance* (respectively, a *global offensive alliance*) if for every $v \in S$, $|N[v] \cap S| \geq |N[v] - S|$ (respectively, if for every $v \in V - S$, $|N(v) \cap S| \geq |N(v) - S|$). The *global defensive alliance number* $\gamma_a(G)$ (respectively, *global offensive alliance number* $\gamma_o(G)$) is the minimum cardinality of a global defensive alliance (respectively, global offensive alliance) of G. The entire vertex set is both a global defensive and offensive alliances for any graph G, so every graph G has a global defensive (offensive) alliance number. We abbreviate global offensive alliance as goa and global defensive alliance as gda. A goa with minimum cardinality $\gamma_o(G)$ is called a $\gamma_o(G)$-set and likewise for the other sets.

In this note we present two bounds on the global offensive alliance number in terms of the domination and global defensive alliance numbers. More precisely we show that if T is a tree of order n, then $\gamma_o(T) \leq 2\gamma(T) - 1$ and if $n \geq 3$, then $\gamma_o(T) \leq \frac{3}{2}\gamma_a(T) - 1$. Moreover, all extremal trees attaining the first bound are characterized.

2. GLOBAL OFFENSIVE ALLIANCE AND DOMINATION

We begin by giving two bounds on the domination and global offensive alliance numbers that will be useful here.

Theorem 2.1 (Lemańska [7]). *For every tree T of order n with ℓ leaves, $\gamma(T) \geq (n - \ell + 2)/3$.*

Theorem 2.2 (Chellali [1]). *For every bipartite graph G without isolated vertices, $\gamma_o(G) \leq (n - \ell + s)/2$.*

Theorem 2.3. *For every tree T of order $n \geq 3$, $\gamma_o(T) \leq \frac{3}{2}\gamma(T) + \frac{s-2}{2}$.*

Proof. By Theorems 2.2 and 2.1 we have

$$\gamma_o(G) \leq (n - \ell + s)/2 = \frac{3}{2}\Big(\frac{n - \ell + 2}{3} + \frac{s - 2}{3}\Big) \leq \frac{3}{2}\gamma(T) + \frac{s - 2}{2}. \qquad \square$$

Using the fact that every $\gamma(T)$-set contains at least $|S(T)| = s$ vertices, Theorem 2.3 leads immediately to the following two corollaries that can be extended to trees of order one and two. Recall that the *independent domination number* $i(G)$ of a graph G is the minimum cardinality of a set of vertices that is both dominating and independent.

Corollary 2.4. *For every tree T, then $\gamma_o(T) \leq 2\gamma(T) - 1$.*

Corollary 2.5 (Favaron [6]). *For every tree T, $\gamma_o(T) \leq 2i(T) - 1$.*

For the purpose of characterizing the extremal trees attaining the bound in Corollary 2.4, we define the family $\mathcal{G}$ of all trees of order at least three that can be obtained from r disjoint stars by first adding $r - 1$ edges so that they are incident only with centers of the stars and the resulting graph is connected, and then subdividing each new edge exactly twice.

Theorem 2.6. *Let T be a tree. Then $\gamma_o(T) = 2\gamma(T) - 1$ if and only if $T \in \{P_1, P_2\}$ or $T \in \mathcal{G}$.*

Proof. Let T be a tree with $\gamma_o(T) = 2\gamma(T) - 1$. If $\gamma_o(T) = 1$, then $T = P_1$ or P_2 or T is a star of order at least three that belongs to $\mathcal{G}$. Thus we assume that $\gamma_o(T) \geq 2$. Then $n \geq 4$, and by Theorem 2.3, $2\gamma(T) - 1 = \gamma_o(T) \leq \frac{3}{2}\gamma(T) + \frac{s-2}{2}$ implying that $s \geq \gamma(T) \geq s$ and so $\gamma(T) = s$. Since for every tree T of order at least three, there is a $\gamma(T)$-set that contains no leaf we conclude that $S(T)$ is a $\gamma(T)$-set, that is $N[S(T)] = V(T)$. Assume now that T contains two support vertices u and v such that either u, v are adjacent or there is a vertex w adjacent to u and v. Let T_u, T_v be the subtrees resulting by removing the edge uv (if the first case occurs) or the edge uw (if the later case occurs), where $u \in V(T_u)$ and $v \in V(T_v)$. Clearly each of T_u and T_v has order at least two. Let D_u be a $\gamma_o(T_u)$-set and likewise D_v a $\gamma_o(T_v)$-set. Without loss of generality $u \in D_u, v \in D_v$, and so $D_u \cup D_v$ is a goa of T. It follows that $\gamma_o(T) \leq \gamma_o(T_u) + \gamma_o(T_v)$. Also since there is a $\gamma(T)$-set that contains all support vertices, we have $\gamma(T_u) + \gamma(T_v) \leq \gamma(T)$. By Corollary 2.4 we obtain

$$\begin{aligned}\gamma_o(T) &\leq \gamma_o(T_u) + \gamma_o(T_v) \leq 2\gamma(T_u) - 1 + 2\gamma(T_v) - 1 \leq \\ &\leq 2\gamma(T) - 2 < 2\gamma(T) - 1,\end{aligned}$$

a contradiction. Hence no two support vertices are adjacent and every vertex of T is either a support vertex or adjacent to exactly one support vertex. Now let $Q = V(T) - (S(T) \cup L(T))$. Then $G[Q]$ is a forest with no isolated vertices containing exactly $|S(T)| - 1$ edges. Hence $|Q| \leq 2(|S(T)| - 1)$. Let D be a minimum goa of $G[Q]$. Then by Theorem 2.2, $|D| \leq \frac{|Q|}{2} \leq |S(T)| - 1$. Also $D \cup S(T)$ is a goa of T and so

$$2\,|S(T)| - 1 = \gamma_o(T) \leq |D \cup S(T)| \leq |S(T)| + |S(T)| - 1 = 2\,|S(T)| - 1.$$

Consequently $|D| = |S(T)| - 1$ and $|Q| = 2(|S(T)| - 1)$. Thus each component of $G[Q]$ is a path P_2, and so every vertex of Q has degree two in T. We conclude that $T \in \mathcal{G}$.

Conversely, if $T \in \{P_1, P_2\}$, then $\gamma_o(T) = 2\gamma(T) - 1$. If $T \in \mathcal{G}$, then $\gamma(T) = |S(T)|$, and the set of all support vertices plus one vertex incident with each new edge forms a minimum goa of T of size $2\,|S(T)| - 1 = 2\gamma(T) - 1$. □

3. GLOBAL OFFENSIVE AND DEFENSIVE ALLIANCES

We first note that $\gamma_o(T)$ may be larger or smaller than $\gamma_a(T)$ for an arbitrary tree T. Indeed for a double star $S_{p,q}$ with $p, q \geq 3$ we have $\gamma_a(S_{p,q}) = \lceil \frac{p}{2} \rceil + \lceil \frac{q}{2} \rceil > \gamma_o(S_{p,q}) = 2$. For the tree T obtained from two paths P_4 by adding an edge joining two support vertices, and subdividing the new edge exactly twice we have $\gamma_a(T) = 4 < \gamma_o(T) = 5$.

We prove next that for every tree T of order $n \geq 3$, $\gamma_o(T)$ is bounded above by $\frac{3}{2}\gamma_a(T) - 1$.

Theorem 3.1. *For every tree T of order $n \geq 3$, $\gamma_o(T) \leq \frac{3}{2}\gamma_a(T) - 1$.*

Proof. We proceed by induction on the order of T. It is easy to check that if $n = 3$ or 4, then the result is valid. Assume that for every tree T' of order $3 \leq n' < n$, we have $\gamma_o(T') \leq \frac{3}{2}\gamma_a(T') - 1$. Let T be a tree of order n. Since stars and double stars satisfy the result we may assume that T has diameter at least four.

We root T at a leaf r of a maximum eccentricity, and let u be a support vertex at distance $\operatorname{diam}(T) - 1$ from r. Let v be the parent of u and w the parent of v in the rooted tree. Clearly $w \neq r$ since $\operatorname{diam}(T) \geq 4$. Let S be any $\gamma_a(T)$-set such that S contains the fewest number of leaves possible. Since every child of v is either a leaf or a support vertex and with our choice of S, it follows that $u, v \in S$ and S contains $\lceil |L_u|/2 \rceil - 1$ leaves of L_u. Consider the following two cases.

Case 1. $|N[v] \cap S| > |N[v] \cap (V(T) - S)|$. Let $T' = T - \{u\} \cup L_u$. Clearly then $S \cap V(T')$ is a gda of T' and hence $\gamma_a(T') \leq \gamma_a(T) - \lceil |L_u|/2 \rceil$. Also if D is any $\gamma_o(T')$-set of T', then $D \cup \{u\}$ is a goa of T, implying that $\gamma_o(T) \leq \gamma_o(T') + 1$. By induction on T' we obtain

$$\gamma_o(T) - 1 \leq \gamma_o(T') \leq \frac{3}{2}\gamma_a(T') - 1 \leq \frac{3}{2}(\gamma_a(T) - \lceil |L_u|/2 \rceil) - 1$$

and therefore $\gamma_o(T) < \frac{3}{2}\gamma_a(T) - 1$.

Case 2. $|N[v] \cap S| = |N[v] \cap (V(T) - S)|$. It follows that v is a support vertex of T. We consider two subcases:

Subcase 2.1. $|L_v| \geq 2$. Let v' be any leaf neighbor of v that does not belong to S and let $T' = T - \{u, v'\} \cup L_u$. Then $S \cap V(T')$ is a gda of T' and so $\gamma_a(T') \leq \gamma_a(T) - \lceil |L_u|/2 \rceil$. Also $\gamma_o(T) \leq \gamma_o(T') + 1$ since every $\gamma_o(T')$-set of T' can be extended to a goa of T by adding u. Now by using the induction on T' we obtain $\gamma_o(T) < \frac{3}{2}\gamma_a(T) - 1$.

Subcase 2.2. $|L_v| = 1$. Then the facts $v \in S$ and $|N[v] \cap S| = |N[v] \cap (V(T) - S)|$ imply that $\deg_T(v) = 3$ and $w \notin S$. Let v' be the unique leaf adjacent to v. We first suppose that $\deg_T(w) \geq 3$. If w is dominated by at least two vertices of S, then let $T' = T - T_v$. Clearly $S \cap V(T')$ is a gda of T' and so $\gamma_a(T') \leq \gamma_a(T) - \lceil |L_u|/2 \rceil - 1$. Also if D is any $\gamma_o(T')$-set of T', then $D \cup \{u, v\}$ is a goa of T, implying that $\gamma_o(T) \leq \gamma_o(T') + 2$. By induction on T' we obtain

$$\gamma_o(T) - 2 \leq \gamma_o(T') \leq \frac{3}{2}\gamma_a(T') - 1 \leq \frac{3}{2}(\gamma_a(T) - \lceil |L_u|/2 \rceil - 1) - 1$$

and therefore $\gamma_o(T) < \frac{3}{2}\gamma_a(T) - 1$. Thus we assume that v is the unique neighbor of w in S. Hence w is not a support vertex. Suppose that a is a leaf in T_w at distance three from w with $a \notin L_u$ and let a-b-c-w be the unique path from a to w. Then according to the previous (sub)cases and by our choice of S vertices b and c play the same role as u and v, respectively, and so b, c belong to S which contradicts the fact that $N(w) \cap S = \{v\}$. Thus every leaf in $T_w - T_v$ is at distance two from w. Furthermore all such leaves are in S since $N(w) \cap S = \{v\}$. It follows that all children of w except v are support vertices of degree two. Let $T' = T - T_w$, $Q \cup \{v\}$ be the set of children of w and $L(Q)$ the set of leaves adjacent to Q. Note that $|Q| = |L(Q)|$. Since T is rooted at a leaf r of maximum eccentricity, T' is nontrivial. If $|V(T')| = 2$, then it can be seen that $\gamma_o(T) = 3 + |Q|$, $\gamma_a(T) \geq 3 + |L(Q)|$ and the result is valid. Hence we assume that $|V(T')| \geq 3$. Then $S \cap V(T')$ is a gda of T' and so

$\gamma_a(T') \le \gamma_a(T) - \lceil |L_u|/2 \rceil - 1 - |L(Q)|$. On the other hand if D is any $\gamma_o(T')$-set of T', then $D \cup Q \cup \{u, v\}$ is a goa of T, implying that $\gamma_o(T) \le \gamma_o(T') + 2 + |Q|$. By induction on T' we obtain

$$\gamma_o(T) - 2 - |Q| \le \gamma_o(T') \le \frac{3}{2}\gamma_a(T') - 1 \le \frac{3}{2}(\gamma_a(T) - \lceil |L_u|/2 \rceil - 1 - |L(Q)|) - 1$$

and therefore $\gamma_o(T) < \frac{3}{2}\gamma_a(T) - 1$.

Finally suppose that $\deg_T(w) = 2$. Let $T' = T - T_w$. If T' has order one, then $r \in S$ and $\{w\} \cup S - \{r\}$ is a $\gamma_a(T)$-set with less leaves than S, a contradiction. If $|V(T')| = 2$, then it can be seen that $\gamma_o(T) = 3$, $\gamma_a(T) \ge 3$, and so $\gamma_o(T) < \frac{3}{2}\gamma_a(T) - 1$. Thus we assume that $|V(T')| \ge 3$. Then $S \cap V(T')$ is a gda of T' and so $\gamma_a(T') \le \gamma_a(T) - \lceil |L_u|/2 \rceil$. If D is any $\gamma_o(T')$-set of T', then $D \cup \{u, v, w\}$ is a goa of T, implying that $\gamma_o(T) \le \gamma_o(T') + 3$. By induction on T' we obtain

$$\gamma_o(T) - 3 \le \gamma_o(T') \le \frac{3}{2}\gamma_a(T') - 1 \le \frac{3}{2}(\gamma_a(T) - \lceil |L_u|/2 \rceil - 1) - 1$$

and therefore $\gamma_o(T) \le \frac{3}{2}\gamma_a(T) - 1$. This completes the proof of the theorem. □

We note that $(2\gamma(T) - 1)$ and $\left(\frac{3}{2}\gamma_a(T) - 1\right)$ are incomparable. First, the difference $(2\gamma(T) - 1) - \left(\frac{3}{2}\gamma_a(T) - 1\right)$ can be arbitrarily large even when $\gamma_o(T) = \frac{3}{2}\gamma_a(T) - 1$ as can be seen with the following tree R_k. Let R_k, $k \ge 0$, be the tree formed from a path on $4k + 6$ vertices labeled $v_1, v_2, \ldots, v_{4k+6}$ by attaching a path of length one to each vertex labeled v_i where $i \equiv 1$ or $2 (mod\ 4)$. For example, the tree R_1 is illustrated in Figure 1. Then $\gamma(R_k) = \gamma_a(R_4) = 2k + 4$, $\gamma_o(R_k) = 3k + 5$, and $\frac{3}{2}\gamma_a(R_k) - 1 = \gamma_o(R_k) = 3k + 5 < 2\gamma(R_k) - 1 = 4k + 7$.

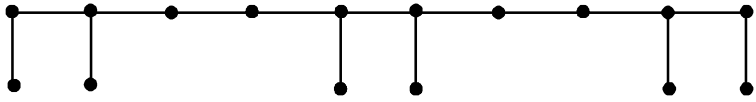

Fig. 1. The tree R_1

On the other hand, the difference $\left(\frac{3}{2}\gamma_a(T) - 1\right) - (2\gamma(T) - 1)$ can also be arbitrarily large even when $\gamma_o(T) = 2\gamma(T) - 1$. To see consider a star $K_{1,k}$ for $k \ge 3$. Then $\gamma_o(K_{1,k}) = \gamma(K_{1,k}) = 1$, $\gamma_a(K_{1,k}) = 1 + \left\lfloor \frac{k}{2} \right\rfloor$ and $\gamma_o(K_{1,k}) = 2\gamma(K_{1,k}) - 1 < \frac{3}{2}\gamma_a(K_{1,k}) - 1 = \frac{1}{2} + \frac{3}{2}\left\lfloor \frac{k}{2} \right\rfloor$.

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