SOME CONSTRUCTIONS OF LYAPUNOV FUNCTION FOR LINEAR EXTENSIONS OF DYNAMICAL SYSTEMS

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Abstract. In this note we consider some sets of linear extensions of dynamical systems and research regularity by means of the sign-changing Lyapunov function. We examine some constructions of Lyapunov functions for given systems.

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1. INTRODUCTION

Let us consider a system of differential equations

$$\frac{dx}{dt} = f(x), \quad \frac{dy}{dt} = A(x)y, \tag{1.1}$$

where $x = (x_1, \dots, x_m) \in T_m$, $y \in \mathbb{R}^n$, $f(x) \in C_{Lip}(T_m)$, a square n-dimensional matrix $A(x) \in C^0(T_m)$, T_m – m-dimensional torus, $C^0(T_m)$ – a space of continuous real functions $F(x) = F(x_1, \dots, x_m)$ which are periodic with period 2π with respect to each variable x_j, $j = \overline{1, m}$, it means it is specified on a torus T_m, $C_{Lip}(T_m)$ – a space of functions which satisfy Lipschitz inequality.

In the literature systems (1.1) are called linear extensions of dynamical systems on a torus [1–7].

When we replace the Cauchy problem solution $x(t;x)$: $\frac{dx}{dt} = f(x), x\big|_{t=0} = x$, into the second equation of (1.1) we obtain a linear system of differential equations $\frac{dy}{dt} = A(x(t;x))y$ with respect to $x \in T_m$. Let $\Omega^t_\tau(x)$ be a normalized fundamental matrix of the system, $\Omega^t_\tau(x)\big|_{t=\tau} = I_n$, I_n is an n-dimensional identical matrix. $\langle y, \tilde{y} \rangle = \sum_{i=1}^n y_i \tilde{y}_i$ – an inner product in $\mathbb{R}^n$. The norm of $n \times n$-dimensional matrix A we denote as $\|A\| = \max_{\|y\|=1} \|Ay\|$, $\|y\| = \sqrt{\langle y, y \rangle}$. $C'(T_m; f)$ – a subspace of $C^0(T_m)$ which consists of functions $F(x)$ such that $F(x(t;x))$ is a continuously differentiable function with respect to $t \in \mathbb{R}$. Moreover $\dot{F}(x) =^{df} \frac{dF(x(t;x))}{dt}\big|_{t=0}$, $\dot{F}(x) \in C^0(T_m)$.

Definition 1.1. Let $C(x)$ be an $n \times n$-dimensional matrix whose elements are real, continuous functions defined on an m – dimensional torus T_m, such that function $G_0(\tau, x)$

$$G_0(\tau, x) = \begin{cases} \Omega^0_\tau(x) C(x(\tau, x)), & \tau \leq 0, \\ \Omega^0_\tau(x) \left[C(x(\tau, x)) - I_n \right], & \tau > 0, \end{cases} \tag{1.2}$$

satisfies the estimate

$$\|G_0(\tau, x)\| \leq K e^{-\gamma |\tau|}$$

with positive constants K and γ, which do not depend on $x \in T_m$ and $\tau \in \mathbb{R}$. Then the function (1.2) is called the Green function of the invariant torus of system (1.1).

In the case when the Green function (1.2) is unique the system (1.1) is called *regular*. Otherwise when the system (1.1) possesses many different Green functions (1.2), the system (1.1) is called *sharply-weak regular*.

Existence of Green function of linear extensions of dynamical systems for invariant torus is considered in [5–7].

The regularity of systems (1.1) can be examined by means of the Lyapunov function [1, 4]. Let us consider Lyapunov functions as a square form

$$V(x, y) = \langle S(x) y, y \rangle, \tag{1.3}$$

where the symmetric matrix $S(x) \in C^1(T_m)$, $C^1(T_m) \subset C^0(T_m)$. Let us notice, that the Lyapunov function is defined to be sign-changing. It means that the Lyapunov function (the square form (1.3)) can change sign, and the derivative $\dot{V}$ with respect to system (1.1) is positive (or negative) definite

$$\dot{V} = \left\langle \left[\frac{\partial S(x)}{\partial x} f(x) + S(x) A(x) + A^T(x) S(x) \right] y, y \right\rangle \geq \|y\|^2.$$

It is clear (see [1]) that system (1.1) is regular when the square form (1.3) exists. Moreover, the derivative with respect to system (1.1) is positive definite, where the symmetric matrix $S(x)$ is nondegenerated

$$\det S(x) \neq 0 \quad \text{for all } x \in T_m.$$

Let W be a square form

$$W = \left\langle \bar{S}(x) z, z \right\rangle, \quad z \in \mathbb{R}^n, \tag{1.4}$$

with matrix of coefficients $\bar{S}(x) = -S^{-1}(x)$. It is readily verified that the derivative of (1.4) with respect to the adjoint system to (1.1)

$$\frac{dx}{dt} = f(x), \quad \frac{dz}{dt} = -A^T(x) z, \quad z \in \mathbb{R}^n, \tag{1.5}$$

is positive definite

$$\dot{W} = \left\langle \left[\frac{\partial \bar{S}(x)}{\partial x} f(x) - \bar{S}(x) A^T(x) - A(x) \bar{S}(x) \right] z, z \right\rangle \geq \epsilon \|z\|^2, \quad \epsilon = \text{const} > 0. \tag{1.6}$$

When system (1.1) is regular it means that system (1.5) is also regular.

Assume the square form (1.4) exists. The symmetric matrix of coefficients $\bar{S}(x) \in C^1(T_m)$ and for some $x = x_0 \in T_m$ the form vanishes $\det \bar{S}(x_0) = 0$. Moreover the derivative of (1.4) with respect to system (1.5) satisfies the inequality (1.6). This shows that the Green function (1.2) for system (1.1) is not unique and the Green function for system (1.5) does not exist.

Let system (1.1) be *regular* which means the Green function (1.2) is *unique*. It is clear (see [1]) that $C(x)$ is the projection matrix

$$C^2(x) \equiv C(x) \quad \text{for all } x \in T_m, \tag{1.7}$$

which fulfills the following identity

$$C(x(\tau;x)) \equiv \Omega_0^\tau(x) C(x) \Omega_\tau^0(x) \quad \text{for all } \tau \in \mathbb{R},\ x \in T_m. \tag{1.8}$$

In this case the Lyapunov function for the system (1.1) can be taken in the following square form

$$V = \langle (S_1(x) - S_2(x))\, y, y \rangle, \quad y \in \mathbb{R}^n,$$

where

$$S_1(x) = \int\limits_{-\infty}^{0} \left\{\Omega_0^t(x)\,[C(x) - I_n]\right\}^T \cdot H(x(t;x)) \cdot \left\{\Omega_0^t(x)\,[C(x) - I_n]\right\} dt,$$

$$S_2(x) = \int\limits_{0}^{+\infty} \left\{\Omega_0^t(x)\,[C(x)]\right\}^T \cdot H(x(t;x)) \cdot \left\{\Omega_0^t(x) C(x)\right\} dt,$$

where $H(x)$ is a symmetric matrix ($H(x) \in C^0(T_m)$), which is positive definite

$$\langle H(x)y, y \rangle \geq h\|y\|^2, \quad h = \text{const} > 0$$

or negative definite

$$\langle H(x)y, y \rangle \leq -h\|y\|^2, \quad h = \text{const} > 0.$$

If the system (1.1) possesses more than one Green function (1.2), then identities (1.7) and (1.8) are not met and the Lyapunov function exists only for system (1.5). Some such functions are of the form

$$W = \left\langle \left(\bar{S}_1(x) - \bar{S}_2(x)\right) z, z \right\rangle,$$

where

$$\bar{S}_1(x) = \int\limits_{-\infty}^{0} \left\{\Omega_\tau^0(x) C(x(\tau;x))\right\} \cdot H_1(x(\tau;x)) \cdot \left\{\Omega_\tau^0 C(x(\tau;x))\right\}^T d\tau,$$

$$\bar{S}_2(x) = \int\limits_{0}^{\infty} \left\{\Omega_\tau^0(x)\,[C(x(\tau;x)) - I_n]\right\} \cdot H_2(x(\tau;x)) \cdot \left\{\Omega_\tau^0\,[C(x(\tau;x)) - I_n]\right\}^T d\tau,$$

where both symmetric matrices $H_i(x) \in C^0(T_m)$, $i = 1, 2$ are positive definite

$$\langle H_i(x)z, z\rangle \geq h\|z\|^2, \quad h = \text{const} > 0$$

or negative definite

$$\langle H_i(x)z, z\rangle \leq -h\|z\|^2, \quad h = \text{const} > 0$$

at the same time. For the case in (1.1) $y \in \mathbb{R}$ and the system is regular, the Lyapunov function (1.3)

$$V = s(x)y^2,$$

where the scalar function $s(x) \in C'(T_m; f)$ can be in one of the following

$$s(x) = \int_{-\infty}^{0} \left(\Omega_0^t(x)\right)^2 H(x(t;x))dt \tag{1.9}$$

or

$$s(x) = \int_{0}^{+\infty} \left(\Omega_0^t(x)\right)^2 H(x(t;x))dt \tag{1.10}$$

and $H(x)$ is any continuous scalar function ($H(x) \in C^0(T_m)$), which fulfills inequality $H(x) > 0$.

Remark 1.2. Let the scalar function $s(x) \in C'(T_m; f)$ be in the form (1.9). Then the inequality

$$\dot{s}(x) + 2s(x)A(x) > 0 \tag{1.11}$$

holds. Moreover, *every* solution of the inequality (1.11) is of the form (1.9). (When $s(x)$ is differentiable, then $\dot{s}(x) = \frac{ds(x)}{dx} f(x)$.)

2. MAIN RESULTS

In [2] there is an equation, which describes the parametric resonance during a crystal modular semiconductor lighting problem. With the use of an asymptotic method the linear extension of the dynamical system converts to linear differential equation. It then follows that it is very interesting to do research in to linear extensions of dynamical systems on a torus, which by means of a change of variables can be converted to system of linear differential equations.

Let

$$\frac{dy_1}{dt} = a_{11}y_1 + a_{12}y_2, \quad \frac{dy_2}{dt} = a_{21}y_1 + a_{22}y_2, \tag{2.1}$$

be a system with constant matrix of coefficients. Changing into polar coordinates $y_1 = y\cos x$, $y_2 = y \sin x$

$$\frac{dx}{dt} = \omega + k_1 \cos 2x - k_2 \sin 2x, \quad \frac{dy}{dt} = [b + k_2 \cos 2x + k_1 \sin 2x]\, y,$$

where $a_{11} = b + k_2$, $a_{12} = k_1 - \omega$, $a_{21} = k_1 + \omega$, $a_{22} = b - k_2$. Let $k_1^2 + k_2^2 > 0$ and

$$k = \sqrt{k_1^2 + k_2^2}, \quad \cos\Delta = \frac{k_1}{k}, \quad x - \frac{\Delta}{2} \to x, \quad kt \to t, \quad \frac{\omega}{k} \to \omega, \quad \frac{b}{k} \to b,$$

then the system (2.1) results

$$\frac{dx}{dt} = \omega - \sin 2x, \quad \frac{dy}{dt} = [b + \cos 2x]y, \tag{2.2}$$

where $\omega, b \in \mathbb{R}$.

System (2.2) is similar to system (1.1). System (2.2) regularity is researched in [3]. However, the Lyapunov function is not constructed in all cases. Therefore, we propose the Lyapunov function construction.

Let

$$\omega^2 + b^2 < 1.$$

Then the system (2.2) does not possess any Green function and the adjoint system:

$$\frac{dx}{dt} = \omega - \sin 2x, \quad \frac{dz}{dt} = -[b + \cos 2x]z, \tag{2.3}$$

possesses more than one Green function. It follows that the derivative to the scalar Lyapunov function

$$V = (\cos 2x)y^2$$

with respect to system (2.2) is positive definite

$$\begin{aligned}\dot{V} &= \{-2\sin 2x\,[\omega - \sin 2x] + 2\cos 2x\,[b + \cos 2x]\}\,y^2 = \\ &= \left\{2 - 2\sqrt{b^2 + \omega^2}\sin(2x + \Delta)\right\}y^2 \geq 2\left(1 - \sqrt{b^2 + \omega^2}\right)y^2.\end{aligned}$$

Moreover, $\cos 2x = 0$ for $x = \frac{\pi}{4} + \frac{\pi n}{2}$, $n \in \mathbb{Z}$.

When parameters (2.2) satisfy the equality

$$\omega^2 + b^2 = 1,$$

we always find such an x_0 such that

$$\begin{aligned}0 &= \omega - \sin 2x_0, \\ 0 &= b + \cos 2x_0.\end{aligned}$$

It then follows that neither of system (2.2) or (2.3) possesses a Green function.

Assume in (2.2) the inequality

$$\omega^2 + b^2 > 1, \quad b \neq 0 \tag{2.4}$$

is met. Research in to system (2.2) (see [3]) shows that, when inequality (2.4) is met, system (2.2) is regular, so the Green function for the system is unique. Our task is to determine the Lyapunov function for the system in four cases.

I. Let $b^2 > 1$. The Lyapunov function for the system is $V = y^2$.

II. In the second case let $|\omega| > 1$, $b \neq 0$. The Lyapunov function construction for (2.2) is identical to the Lyapunov function construction to the following system:

$$\frac{dx}{dt} = 1, \quad \frac{dy}{dt} = \frac{b + \cos 2x}{\omega - \sin 2x} y.$$

For a homogeneous linear equation with parameter x

$$\frac{dy}{dt} = \frac{b + \cos 2(t+x)}{\omega - \sin 2(t+x)} y$$

let us find a function $\Omega_\tau^t(x)$

$$\Omega_\tau^t(x) = \exp\left\{ \int_\tau^t \frac{b}{\omega - \sin 2(\sigma + x)} d\sigma \right\} \cdot F(t, \tau, x),$$

where

$$F(t, \tau, x) = \exp\left\{ \int_\tau^t \frac{\cos 2(\sigma + x)}{\omega - \sin 2(\sigma + x)} d\sigma \right\} = \left(\frac{\omega - \sin 2(\tau + x)}{\omega - \sin(t + x)} \right)^{\frac{1}{2}}.$$

The form of (1.9) and (1.10) implies that

$$s(x) = \int_{-\infty}^0 \exp\left\{ 2\int_0^t \frac{b}{\omega - \sin 2(\sigma + x)} d\sigma \right\} \left(\frac{\omega - \sin 2x}{\omega - \sin 2(t + x)} \right) H(x(t;x))dt, \quad \frac{b}{\omega} > 0,$$

$$s(x) = \int_0^\infty \exp\left\{ 2\int_0^t \frac{b}{\omega - \sin 2(\sigma + x)} d\sigma \right\} \left(\frac{\omega - \sin 2x}{\omega - \sin 2(t + x)} \right) H(x(t;x))dt, \quad \frac{b}{\omega} < 0.$$

Since the integral

$$\int_0^t \frac{b}{\omega - \sin 2(\sigma + x)} d\sigma$$

has form

$$\int_0^t \frac{b}{\omega - \sin 2(\sigma + x)} d\sigma = \alpha t + \Phi(t, x),$$

where

$$\alpha = \int_0^\pi \frac{b}{\omega - \sin 2(\sigma + x)} d\sigma = \int_x^{\pi + x} \frac{b}{\omega - \sin 2z} dz = \text{const}$$

and the function

$$\Phi(t,x) = \int_0^t \left[\frac{b}{\omega - \sin 2(\sigma + x)} - \alpha\right] d\sigma$$

is bounded, then the scalar function $s(x)$ which fulfills one of the inequalities

$$\frac{ds(x)}{dx}(\omega - \sin 2x) + 2s(x)(b + \cos 2x) > 0 \tag{2.5}$$

or

$$\frac{ds(x)}{dx}(\omega - \sin 2x) + 2s(x)(b + \cos 2x) < 0, \tag{2.6}$$

can be in the form

$$s(x) = \omega - \sin 2x.$$

Let us check

$$\frac{ds(x)}{dx}(\omega - \sin 2x) + 2s(x)(b + \cos 2x) = -2\cos 2x(\omega - \sin 2x) +$$

$$+2(\omega - \sin 2x)(b + \cos 2x) = 2b(\omega - \sin 2x).$$

III. Let $\omega^2 = 1$, $b \neq 0$. From [3] we know that the Green function exists in cases III and IV, but the Lyapunov function is not determined. In the case

$$\Omega_0^t(x) = e^{bt} \cdot \begin{cases} \sqrt{2t^2(\cos x - \sin x)^2 + 2t\cos 2x + 1}, & \omega = 1, \\ \sqrt{2t^2(\cos x + \sin x)^2 + 2t\cos 2x + 1}, & \omega = -1. \end{cases}$$

When the conditions

$$\omega = 1, \quad 0 < b < 1 \tag{2.7}$$

are met, then

$$\left(\Omega_0^t(x)\right)^2 = 2te^{2bt}\cos 2x - 2t^2 e^{2bt}\sin 2x + (2t^2 + 1)e^{2bt}.$$

On the basis of equality (1.9) the form of $s(x)$ is given by

$$s(x) = \int_{-\infty}^0 \left[2te^{2bt}\cos 2x - 2t^2 e^{2bt}\sin 2x + (2t^2 + 1)e^{2bt}\right] H(x(t;x))dt.$$

It then follows that, when conditions (2.7) are met, the function $s(x)$ which fulfills the inequality (2.5) can be in the form

$$s(x) = L\cos 2x + M\sin 2x + N, \tag{2.8}$$

where L, M, N are constant. As function (2.8) has only positive values, the inequality

$$\sqrt{L^2 + M^2} < N \tag{2.9}$$

holds. To determine constants L, M, N let us put function (2.8) to the left side of the inequality (2.5)

$$\frac{ds(x)}{dx}(\omega - \sin 2x) + 2s(x)(b + \cos 2x) = 2(Lb + M + N)\cos 2x +$$

$$+2(Mb - L)\sin 2x + 2L + 2Nb.$$

The inequality (2.5) holds, when

$$\begin{cases} L + Mb > 0, \\ (Lb + M + N)^2 + (Mb - L)^2 < (L + Nb)^2. \end{cases} \tag{2.10}$$

The second inequality of (2.10) implies that

$$L^2b^2 + M^2(1 + b^2) + N^2(1 - b^2) + 2MN < 0. \tag{2.11}$$

When (2.7) holds, a necessary condition for (2.6) is $M < 0$. Let $L = 0$ and the inequality (2.11) has the form

$$M^2(1 + b^2) + N^2(1 - b^2) < 2(-M)N.$$

It then follows that

$$\frac{-M}{N}(1 + b^2) + \frac{N}{-M}(1 - b^2) < 2. \tag{2.12}$$

Let us prove that there exist $M < 0$, $N > 0$ for which the inequality (2.12) holds. Let us put

$$x = \frac{-M}{N}$$

and study the function

$$f(x) = x(1 + b^2) + \frac{1 - b^2}{x}, \quad x > 0. \tag{2.13}$$

The minimal value of the function (2.13) in

$$x = \sqrt{\frac{1 - b^2}{1 + b^2}} = x_\mu < 1$$

has the value

$$f_{min} = f(x_\mu) = 2\sqrt{1 - b^4} + 2.$$

Therefore, for every fixed $0 < b < 1$ there exist $M < 0$ and $N > 0$ such that for $L = 0$ inequalities (2.9) and (2.10) hold. Let $M = -1$. Then the function

$$s(x) = \sqrt{\frac{1 + b^2}{1 - b^2}} - \sin 2x$$

fulfills (2.5).

IV. Let $\omega^2 < 1$, $b^2 < 1$, $\omega^2 + b^2 > 1$. It this case

$$\Omega_0^t = \exp\left\{\left(b - \frac{\gamma}{2}\right) t\right\} \sqrt{\frac{u^2 e^{2\gamma t} - 2Buve^{\gamma t} + C\nu^2}{u^2 - 2Buv + Cv^2}}, \tag{2.14}$$

where

$$B = \frac{\omega^2}{1 - \sqrt{1 - \omega^2}}, \quad C = \frac{1 + \sqrt{1 - \omega^2}}{1 - \sqrt{1 - \omega^2}}, \tag{2.15}$$

$$\begin{aligned} u &= \omega \cdot \sin x - (1 + \sqrt{1 - \omega^2}) \cos x, \\ \nu &= \omega \cdot \sin x + (-1 + \sqrt{1 - \omega^2}) \cos x, \quad \gamma = 2\sqrt{1 - \omega^2}. \end{aligned} \tag{2.16}$$

This implies that the change of variables (2.15) and (2.16) in (2.14) gives

$$u^2 - 2Buv + Cv^2 \equiv \text{const}.$$

Indeed,

$$u^2 = \omega^2 \sin^2 x + \left(1 + \sqrt{1 - \omega^2}\right)^2 \cos^2 x - 2\omega \left(1 + \sqrt{1 - \omega^2}\right) \sin x \cos x,$$

$$Cv^2 = \tfrac{1+\sqrt{1-\omega^2}}{1-\sqrt{1-\omega^2}} \cdot$$

$$\cdot \left[\omega^2 \sin^2 x + (-1 + \sqrt{1 - \omega^2})^2 \cos^2 x + 2\omega \left(-1 + \sqrt{1 - \omega^2}\right) \sin x \cos x\right],$$

$$-2Buv = -2\tfrac{\omega^2}{1-\sqrt{1-\omega^2}} \left[\omega^2 - 2\omega \sin x \cos x\right]$$

and

$$u^2 - 2Buv + Cv^2 \equiv 2\left(1 + \sqrt{1 - \omega^2}\right)\left(1 - \omega^2\right).$$

From (2.14), when (2.15) and (2.16) holds, the scalar function $s(x)$ which satisfies the inequality

$$\frac{ds(x)}{dx}(\omega - \sin 2x) + 2s(x)(b + \cos 2x) > 0 \tag{2.17}$$

for ω, b

$$0 < \omega < 1, \quad b^2 < 1, \quad \omega^2 + b^2 > 1$$

can be taken in the form (2.8). Let us put (2.8) into the left side of inequality (2.17)

$$\frac{ds(x)}{dx}(\omega - \sin 2x) + 2s(x)(b + \cos 2x) = 2(Lb + M\omega + N) \cos 2x +$$

$$+ 2(Mb - L\omega) \sin 2x + 2L + 2Nb.$$

The right side of inequality (2.17) is positive, when the inequalities

$$\begin{cases} L + Nb > 0, \\ (Lb + M\omega + N)^2 + (Mb - L\omega)^2 < (L + Nb)^2, \end{cases} \tag{2.18}$$

hold. It then follows that

$$L^2(b^2+\omega^2-1)+M^2(\omega^2+b^2)+N^2(1-b^2)+2MN\omega<0. \tag{2.19}$$

Let $L=0$, $M<0$, $N>0$. Then inequality (2.19) implies

$$-\frac{M}{N}(\omega^2+b^2)+\frac{N}{-M}(1-b^2)<2\omega. \tag{2.20}$$

There exist $M<0$, $N>0$ such that inequality (2.20) holds for every fixed $b\in(0,1)$. Let

$$x=\frac{-M}{N}$$

and the function

$$f(x)=x(\omega^2+b^2)+\frac{1-b^2}{x}, \quad x>0$$

in

$$x=\sqrt{\frac{1-b^2}{\omega^2+b^2}}=x_\nu$$

has a minimal value

$$f_{min}=f(x_\nu)=2\sqrt{(1-b^2)(\omega^2+b^2)}.$$

It is easy to see that $f_{min}<2\omega$. Hence, for every fixed $b\in[-1,0)\cup(0,1]$ there exist $M<0$, $N>0$ such that $L=0$ and inequalities (2.9) and (2.18) hold. Let $M=-1$ and the function $s(x)$ be given by

$$s(x)=\sqrt{\frac{\omega^2+b^2}{1-b^2}}-\sin 2x.$$

The result can be summarized as follows.

Proposition 2.1. *The Lyapunov function for system* (2.2) *has the following form:*
I. *If* $|b|>1$

$$V=y^2.$$

II. *If* $|\omega|>1$, $b\neq 0$

$$V=(\omega-\sin 2x)y^2.$$

III. *If* $\omega^2\leq 1$, $b^2<1$, $\omega^2+b^2>1$

$$V=(\lambda-\sin 2x)\,y^2, \quad \begin{cases}\lambda=\sqrt{\frac{\omega^2+b^2}{1-b^2}}, & \omega\in(0,1\rangle,\\ \lambda=-\sqrt{\frac{\omega^2+b^2}{1-b^2}}, & \omega\in\langle-1,0).\end{cases}$$

IV. *If* $b^2=1$, $0<|\omega|\leq 1$

$$V=(\lambda-\sin 2x)y^2, \quad \begin{cases}\lambda>\frac{\omega^2+1}{2\omega}, & 0<\omega\leq 1,\\ \lambda<\frac{\omega^2+1}{2\omega}, & -1<\omega<0.\end{cases}$$

REFERENCES

[1] Yu.A. Mitropolsky, A.M. Samoilenko, V.L. Kulik, *Dichotomies and Stability in Nonautonomous Linear Systems*, Taylor&Francis Inc., London, 2003.

[2] B.I. Moseenkov, P.A. Maksimyuk, *Studying parametric resonance when semiconductor crystals are illuminated by modulated light*, Naukova Dumka, Kiev, 1977 [in Russian].

[3] N.W. Stiepanenko, E. Tkocz-Piszczek, *Problems of regularity of linear extensions of dynamical systems on a torus*, Nonlinear Oscil. **12** (2009), 99–109.

[4] V.L. Kulik, N.W. Stiepanenko, *Alternating-sign Lyapunov functions in the theory of linear extensions of dynamical systems on a torus*, Ukrain. Mat. Zh. **59** (2007) 4, 489–500; translation in Ukrainian Math. J. **59** (2007) 4, 546–562 [in Ukrainian].

[5] H.M. Kulik, V.L. Kulik, *Existence of Green-Samoilenko functions of some linear extensions of dynamical systems*, Nonlinear Oscill. **7** (2004), 468–474.

[6] A.A. Boichuk, *A condition for the existence of a unique Green–Samoilenko function for the problem of invariant torus*, Ukrainian Math. J. **53** (2001), 556–559.

[7] A.M. Samoilenko, *On the existence of a unique Green function for the linear extension of a dynamical system on a torus*, Ukrainian Math. J. **53** (2001), 513–521.

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