THE FEJER-RIESZ TYPE RESULT FOR SOME WEIGHTED HILBERT SPACES OF ANALYTIC FUNCTIONS IN THE UNIT DISC

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Abstract. In this note we prove Fejer-Riesz inequality type results for some weighted Hilbert spaces of analytic functions in the unit disc. We describe also a class of such spaces for which Fejer-Riesz inequality type results do not hold.

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1. INTRODUCTION

Denote by U the unit disc in the complex plane $\mathbb{C}$. Given $s > -1$ set

$$A^{2,s}(U) = \left\{ f \text{ holomorphic in } U : \int_U |f(z)|^2 (1-|z|^2)^s dm(z) < +\infty \right\}. \tag{1.1}$$

The spaces $A^{2,s}(U)$ were considered by many authors; see e.g. [1, 2, 6–8].

Given a function f holomorphic in U, $f(z) = \sum_{n=0}^{\infty} a_n z^n$, $z \in U$, one can prove by integrating in polar coordinates and using the formula

$$(p+1)^{1+s} \int_0^1 u^p (1-u)^s du \to \Gamma(s+1) \tag{1.2}$$

as $p \to \infty$ (where Γ is Euler's gamma function) that

$$f \in A^{2,s}(U) \text{ iff } \sum_{n=0}^{\infty} \frac{|a_n|^2}{(n+1)^{1+s}} < +\infty. \tag{1.3}$$

In particular for $s = 0$ we have $A^{2,0}(U) = L^2H(U)$, the space of all holomorphic functions in U which are L^2-integrable with respect to the Lebesgue measure in U; this space is called the Bergman space. For $s = 0$ the condition (1.3) has the form

$$\sum_{n=0}^{\infty} \frac{|a_n|^2}{(n+1)} < +\infty.$$

Moreover, for $f(z) = \sum_{n=0}^{\infty} a_n z^n$ holomorphic in U we have

$$\int_U |f(z)|^2 (1-|z|^2)^s dm(z) = \sum_{n=0}^{\infty} |a_n|^2 \int_U |z|^{2n} (1-|z|^2)^s dm(z) =$$

$$= 2\pi \sum_{n=0}^{\infty} |a_n|^2 \int_0^1 t^{2n+1} (1-t^2)^s dt.$$

If $s \leq -1$, for every $n = 0, 1, \ldots,$ it holds

$$\int_0^1 t^{2n+1} (1-t^2)^s dt = +\infty; \tag{1.4}$$

therefore, for $s \leq -1$ the spaces $A^{2,s}(U)$ defined by the integral condition (1.1) consist only of the zero function; on the other hand, the series condition

$$\sum_{n=0}^{\infty} \frac{|a_n|^2}{(n+1)^{1+s}} < +\infty \tag{1.5}$$

in the right-hand side of (1.3) gives non-zero Hilbert spaces of holomorphic functions in U. In this note we will consider such spaces for $s \leq -1$. We recall that for $s = -1$ we obtain the condition $\sum_{n=0}^{\infty} |a_n|^2 < +\infty$, the well – known condition characterizing functions from the Hardy space $H^2(U)$.

For functions from the space $H^2(U)$ it is the Fejer-Riesz inequality is well-known; see e.g. [p. 46][3]; it follows from this that if $f \in H^2(U)$ then for every $z \in \partial U$ it holds

$$\int_0^1 |f(tz)|^2 dt < +\infty.$$

(This inequality is valid for all H^p-spaces with $1 \leq p < +\infty$).

In [5] we have proved a result of similar type for the spaces $A^{2,s-1}(U)$ with $s > 0$:

Proposition 1.1 ([5, Theorem 1]). *Let s be a positive number. Suppose that $f \in A^{2,s-1}(U)$. Then for every $z \in \partial U$*

$$\int_0^1 |f(tz)|^2 (1-t^2)^s dt < +\infty. \tag{1.6}$$

In this note we consider the spaces $A^{2,s}(U)$ with $s \leq -1$ and for some range of the exponent s we prove a result similar to that in Proposition 1.1.

2. THE RESULT

First of all, because of (1.4), for $s \leq -1$ we define the spaces $A^{2,s}(U)$ by the series condition (1.5): For $s \leq -1$ set

$$A^{2,s}(U) = \left\{ f \text{ holomorphic in } U : f(z) = \sum_{n=0}^{\infty} a_n z^n \text{ in } U \text{ and } \sum_{n=0}^{\infty} \frac{|a_n|^2}{(n+1)^{1+s}} < +\infty \right\}. \tag{2.1}$$

Moreover, note that condition (1.6) makes sense for $s > -1$. We want therefore to prove the following result for exponent s with $-2 < s \leq -1$: If s is a given number with $-2 < s \leq -1$, f is holomorphic in U, $f(z) = \sum_{n=0}^{\infty} a_n z^n$, and

$$\sum_{n=0}^{\infty} \frac{|a_n|^2}{(n+1)^{1+s}} < +\infty, \tag{2.2}$$

then for every $z \in \partial U$:

$$\int_0^1 |f(tz)|^2 (1-t^2)^{s+1} dt < +\infty. \tag{2.3}$$

Because of the technical difficulties we are able to prove only a weaker result, which we now describe:

Let $\{b_k\}_{k=0}^{\infty}$ be a new sequence, obtained from $\{a_n\}_{n=0}^{\infty}$ in such a way that we delete all numbers $\{a_n\}$ with $a_n = 0$, and then reorder the remaining numbers a_n to get the new sequence $\{a'_k\}_{k=0}^{\infty}$ with $|a'_0| \geq |a'_1| \geq \dots$, we define then $b_k = |a'_k|$, $k = 0, 1, \dots$.

It is not difficult to prove that also

$$\sum_{n=0}^{\infty} \frac{b_n^2}{(n+1)^{1+s}} < +\infty. \tag{2.4}$$

The additional condition which we assume in order to be able to provide the proof is the following:

$$\text{The sequence } \left\{ \frac{b_n^2}{(n+1)^{1+s}} \right\}_{k=0}^{\infty} \text{ is decreasing.} \tag{2.5}$$

This condition seems to be superfluous for the result described in (2.2) and (2.3) to hold, but as already mentioned we cannot prove (2.3) without assuming it.

Proposition 2.1. *Suppose that $-2 < s \leq -1$; let the function f, holomorphic in U, $f(z) = \sum_{n=0}^{\infty} a_n z^n$ satisfy* (2.2), *and suppose that condition* (2.5) *holds. Then for every $z \in \partial D$*

$$\int_0^1 |f(tz)|^2 (1-t^2)^{s+1} dt < +\infty. \tag{2.6}$$

Proof. It is sufficient to prove that for a given $z \in \partial U$

$$\int_0^1 |f(tz)|^2 (1-t)^{s+1} dt < +\infty. \tag{2.7}$$

We have

$$\begin{aligned}\int_0^1 |f(tz)|^2 (1-t)^{s+1} dt &= \int_0^1 \left| \sum_{n=0}^{\infty} a_n t^n \right|^2 (1-t)^{s+1} dt \leq \\ &\leq \sum_{k,l=0}^{\infty} |a_k||a_l| \int_0^1 (1-t)^{s+1} t^{k+l} dt.\end{aligned} \tag{2.8}$$

By (1.2),

$$(p+1)^{s+2} \int_0^1 (1-t)^{s+1} t^p dt \to \Gamma(s+2)$$

as $p \to \infty$; therefore to prove that the series on the right-hand side of (2.8) is finite it is sufficient to prove that

$$\sum_{k,l=0}^{\infty} |a_k||a_l| \frac{1}{(1+k+l)^{s+2}} \tag{2.9}$$

is convergent. We refer now to the process described in [4, pp. 278–280]. It follows from this that

$$\sum_{k,l=0}^{\infty} \frac{1}{(1+k+l)^{s+2}} |a_k||a_l| \leq \sum_{k,l=0}^{\infty} \frac{1}{(1+k+l)^{s+2}} b_k b_l, \tag{2.10}$$

where $\{b_k\}_{k=0}^{\infty}$ is the sequence defined above; the series in [4], formula (32), differs from our series (2.9), but the reasoning in [4] described on pages 278–280 can be applied to (2.9); the only required property is that the sequence

$$\left\{ \frac{1}{(1+p)^{s+2}} \right\}_{p=0}^{\infty}$$

is decreasing, similarly as the sequence $\{\frac{1}{1+p}\}_{p=0}^{\infty}$ considered in [4].

Therefore, to prove that (2.9) is convergent it is sufficient to prove that the sequence

$$\sum_{k,l=0}^{\infty} \frac{1}{(1+k+l)^{s+2}} b_k b_l$$

on the right-hand side of (2.10) is convergent. Since the sequence $\{b_k\}_{p=0}^{\infty}$ is decreasing we may use the two-sided version of the well-known Cauchy's concentration principle for testing the convergence of the series $\sum_{k=0}^{\infty} c_k$ with $c_0 \geq c_1 \geq c_2 \geq \cdots \geq 0$; this principle says that such a series is convergent iff the series $\sum_{l=0}^{\infty} 2^l c_{2^l}$ is convergent. By applying this two-sided Cauchy's principle to our double series

$$\sum_{k,l=0}^{\infty} \frac{1}{(1+k+l)^{s+2}} b_k b_l$$

we obtain that the series $\sum_{k,l=0}^{\infty} \frac{1}{(1+k+l)^{s+2}} b_k b_l$ is convergent iff the series $\sum_{r,t=0}^{\infty} \frac{2^r 2^t}{(1+2^r+2^t)^{s+2}} b_{2^r} b_{2^t}$ is convergent. Moreover since $(1+2^r+2^t)^{s+2}$ is estimated from below and from above by a constant time of $(2^r)^{s+2}+(2^t)^{s+2}$ it suffices to show that the series

$$\sum_{r,t=0}^{\infty} \frac{2^r 2^t}{(2^r)^{s+2}+(2^t)^{s+2}} b_{2^r} b_{2^t}$$

is convergent. We have

$$\begin{aligned}
&\sum_{r,t=0}^{\infty} \frac{2^r 2^t}{(2^r)^{s+2}+(2^t)^{s+2}} b_{2^r} b_{2^t} = \\
&= \sum_{r,p=0}^{\infty} \frac{2^r 2^{r+p}}{(2^r)^{s+2}+(2^{r+p})^{s+2}} b_{2^r} b_{2^{r+p}} = \\
&= \sum_{r,p=0}^{\infty} \frac{2^{2r} 2^p}{(2^r)^{s+2}(1+(2^p)^{s+2})} b_{2^r} b_{2^{r+p}} \leq \\
&\leq \sum_{r=0}^{\infty}\sum_{p=0}^{\infty} \frac{1}{2^{rs}} b_{2^r} \frac{2^p}{(2^p)^{s+2}} b_{2^{r+p}} \leq \\
&\leq \sum_{r=0}^{\infty}\sum_{p=0}^{\infty} \frac{1}{2^{rs}} b_{2^r} \frac{1}{2^{p(1+s)}} b_{2^{r+p}} = \\
&= \sum_{p=0}^{\infty}\sum_{r=0}^{\infty} \left(\frac{1}{(2^r)^{s/2}} b_{2^r}\right)\left(\frac{1}{(2^r)^{s/2}(2^p)^{s/2}} b_{2^{r+p}}\right) \frac{1}{2^{p(1+s/2)}} \leq \\
&\leq \sum_{p=0}^{\infty} \left(\sum_{r=0}^{\infty}\left(\frac{1}{(2^r)^{s/2}} b_{2^r}\right)^2\right)^{1/2} \left(\sum_{r=0}^{\infty}\left(\frac{1}{(2^{r+p})^{s/2}} b_{2^{r+p}}\right)^2\right)^{1/2} \frac{1}{2^{p(1+s/2)}} \leq \\
&\leq \sum_{p=0}^{\infty} \left(\sum_{r=0}^{\infty} \frac{1}{2^{rs}} b_{2^r}^2\right) \frac{1}{2^{p(1+s/2)}}.
\end{aligned} \tag{2.11}$$

By conditions (2.9) and (2.5) it follows from the aforementioned Cauchy's concentration principle that the series

$$\sum_{r=0}^{\infty} \frac{2^r}{(1+2^r)^{1+s}} b_{2^r}^2$$

is convergent; hence also the series

$$\sum_{r=0}^{\infty} \frac{1}{2^{rs}} b_{2^r}^2$$

is convergent. Therefore the right-hand side of (2.11) is estimated by a constant time of

$$\sum_{p=0}^{\infty} \frac{1}{2^{p(1+s/2)}}.$$

Since $-2 < s \leq -1$, this last sequence as well as both sequences in (2.10) are convergent; this proves that the integral in (2.7) is finite, and so Proposition 2.1 is proved. □

Note that we can prove relatively easily that for f and s like in Proposition 2.1 the integral

$$\int_0^1 |f(tz)|^2 (1-t^2)^{\sigma+1} dt$$

for $\sigma > s$ is convergent, if we assume that f satisfies (2.2); we do not need any further condition like e.g. (2.5).

In fact, using a similar argument to that used in the proof of Proposition 2.1 with s replaced by σ we see that it is sufficient to show that the series

$$\sum_{k,l=0}^{\infty} |a_k||a_l| \frac{1}{(1+k+l)^{\sigma+2}}$$

is convergent. We have by Hölder's inequality

$$\sum_{k,l=0}^{\infty} |a_k||a_l| \frac{1}{(1+k+l)^{\sigma+2}} =$$

$$= \sum_{k,l=0}^{\infty} \frac{|a_k||a_l|}{(k+1)^{(s+1)/2}(l+1)^{(s+1)/2}} \frac{(k+1)^{(s+1)/2}(l+1)^{(s+1)/2}}{(1+k+l)^{\sigma+2}} \leq$$

$$\leq \left(\sum_{k,l=0}^{\infty} \frac{|a_k|^2|a_l|^2}{(k+1)^{s+1}(l+1)^{s+1}} \right)^{1/2} \left(\sum_{k,l=0}^{\infty} \frac{(k+1)^{s+1}(l+1)^{s+1}}{(1+k+l)^{2(\sigma+2)}} \right)^{1/2}.$$

The first series in the right-hand side of the above inequality is convergent by (2.2). As to the second series it is sufficient to show that the series

$$\sum_{k,l=1}^{\infty} \frac{k^{s+1}l^{s+1}}{(k+l)^{2(\sigma+2)}}$$

is convergent. By using the above mentioned Cauchy's concentration principle we see that it remains to show that the series

$$\sum_{r,t=1}^{\infty} \frac{2^r 2^t}{(2^r+2^t)^{2(\sigma+2)}} 2^{r(s+1)} 2^{t(s+1)}$$

is convergent. But

$$\sum_{r,t=1}^{\infty} \frac{2^r 2^t}{(2^r+2^t)^{2(\sigma+2)}} 2^{r(s+1)} 2^{t(s+1)} = \sum_{r,p=1}^{\infty} \frac{2^r 2^{r+p}}{(2^r+2^{r+p})^{2(\sigma+2)}} 2^{r(s+1)} 2^{(r+p)(s+1)}$$

and this last series is a constant time of

$$\sum_{r=1}^{\infty} \left(2^{2(s-\sigma)}\right)^r \sum_{p=1}^{\infty} \left(\frac{1}{2^{2+2\sigma-s}}\right)^p. \tag{2.12}$$

Since $-2 < s < \sigma$, both series in (2.12) converge.

Note that this method does not work for $\sigma = s$, since in that case the first series in (2.12) is divergent.

Now we shall prove that the result of Proposition 2.1 is the best possible in the following sense:

Proposition 2.2. *Let s be a number such that $-2 < s \leq -1$. Then there exists a function $f \in A^{2,s}(U)$ such that for all σ with $-2 < \sigma < s$,*

$$\int_0^1 |f(t)|^2 (1-t^2)^{\sigma+1} dt = +\infty. \tag{2.13}$$

Proof. Like in [5, Theorem 2], set

$$f(z) = \sum_{n=2}^{\infty} \frac{n^{s/2}}{\log n} z^n.$$

We have

$$\sum_{n=2}^{\infty} \frac{|a_n|^2}{(n+1)^{1+s}} = \sum_{n=2}^{\infty} \frac{n^s}{(n+1)^{1+s} \log^2 n}$$

and this last series is convergent, so in virtue of (2.1), $f \in A^{2,s}(U)$.

Moreover, with σ like in the assumption of Proposition 2.2 we have

$$\int_0^1 |f(t)|^2 (1-t^2)^{\sigma+1} dt \approx \int_0^1 |f(t)|^2 (1-t)^{\sigma+1} dt = \sum_{k,l=2}^{\infty} a_k a_l \int_0^1 t^{k+l} (1-t)^{\sigma+1} dt =$$

$$= \sum_{k,l=2}^{\infty} \frac{k^{s/2} l^{s/2}}{\log k \log l} \int_0^1 t^{k+l} (1-t)^{\sigma+1} dt.$$

By virtue of (1.2) instead of the last series we may consider the series

$$\sum_{k,l=2}^{\infty} \frac{k^{s/2} l^{s/2}}{\log k \log l} \frac{1}{(1+k+l)^{2+\sigma}}. \tag{2.14}$$

If τ is any number with $\sigma < \tau < s$, this last series is bounded from below by a constant time of the series

$$\sum_{k,l=2}^{\infty} \frac{k^{s/2} l^{s/2}}{(k+l)^{2+\tau}}. \tag{2.15}$$

To this series we can apply the already mentioned two-sided Cauchy's concentration principle. It follows that our series behaves like the series

$$\sum_{r,t=1}^{\infty} \frac{2^r 2^t 2^{rs/2} 2^{ts/2}}{(2^r + 2^t)^{2+\tau}}. \tag{2.16}$$

But taking the subseries of the last series, consisting only of terms with $r = t$ we obtain

$$\sum_{r=1}^{\infty} \frac{2^{rs} 2^r 2^r}{(2^{r+1})^{2+\tau}} = \frac{1}{2^{2+\tau}} \sum_{r=1}^{\infty} \left(\frac{2^{2+s}}{2^{2+\tau}} \right)^r = \frac{1}{2^{2+\tau}} \sum_{r=1}^{\infty} (2^{s-\tau})^r.$$

Since this last series is divergent, so are the series (2.16), (2.15) and (2.14), and therefore the integral (2.13) is infinite. This ends the proof. □

Finally we want to consider the spaces $A^{2,s}(U)$ with $s \leq -2$. In this case for $f \in A^{2,s}(U)$, $f(z) = \sum_{n=0}^{\infty} a_n z^n$, neither the condition

$$\int_U |f(z)|^2 (1 - |z|^2)^s dm(z) < +\infty \tag{2.17}$$

nor

$$\int_0^1 |f(tz)|^2 (1 - t^2)^{s+1} dt < +\infty, \tag{2.18}$$

$z \in \partial D$, hold. As explained before, the integral condition (2.17) was already replaced by condition

$$\sum_{n=0}^{\infty} \frac{|a_n|^2}{(n+1)^{1+s}} < +\infty. \tag{2.19}$$

Moreover, if the coefficients a_n are non-negative, then it follows from (1.2), (2.8), and (2.9) that for $s > -2$ the expression

$$\int_0^1 |f(tz)|^2 (1 - t^2)^{s+1} dt$$

is estimated from below and from above by a constant time the sum of the series

$$\sum_{k,l=0}^{\infty} |a_k||a_l| \frac{1}{(1+k+l)^{s+2}}.$$

Therefore we can hope that the right analogue of the Fejer-Riesz type result for $s \leq -2$, described already for $-1 \geq s > -2$, would be the following: If f is holomorphic in U, $f(z) = \sum_{n=0}^{\infty} a_n z^n$, and (2.19) holds, then

$$\sum_{k,l=0}^{\infty} |a_k||a_l| \frac{1}{(1+k+l)^{s+2}} < +\infty. \tag{2.20}$$

We show that such a result does not hold for $s \leq -2$; we have namely the following proposition:

Proposition 2.3. *Suppose that s is a real number with $s \leq -2$. Then there exists the function f holomorphic in U, $f(z) = \sum_{n=0}^{\infty} a_n z^n$, such that* (2.19) *holds, but the series in* (2.20) *does not converge.*

Proof. The choice of the function f is very similar to that in Proposition 2.2. Let

$$a_n = \frac{1}{(n+1)^{-s/2} \log n}, \quad n = 2, 3, \ldots.$$

Then

$$\sum_{n=2}^{\infty} \frac{1}{(n+1)^{1+s}} |a_n|^2 = \sum_{n=2}^{\infty} \frac{1}{(n+1) \log^2 n} < +\infty,$$

hence condition (2.19) is satisfied. On the other hand

$$\sum_{k,l=2}^{\infty} a_k a_l \frac{1}{(1+k+l)^{s+2}} = \sum_{k,l=2}^{\infty} \frac{1}{(1+k+l)^{s+2}} \frac{1}{(k+1)^{-s/2} \log k \quad (l+1)^{-s/2} \log l}. \tag{2.21}$$

Consider the subseries of the series in the right-hand side of (2.21), consisting of terms with arbitrary k and with $l = 2$. Then we obtain the series

$$\sum_{k=2}^{\infty} \frac{1}{(k+3)^{s+2} (k+1)^{-s/2} \log k \quad 3^{-s/2} \log 2}. \tag{2.22}$$

Note that the terms in this last series are estimated from below by a constant time of $\frac{1}{k^{(s/2)+2} \log k}$. Since $s \leq -2$ it follows that the series

$$\sum_{k=2}^{\infty} \frac{1}{k^{(s/2)+2} \log k}$$

is divergent, and so also is the series in (2.22), as well as the original series in (2.21). This ends the proof of Proposition 2.3. □

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