STRENGTHENED STONE-WEIERSTRASS TYPE THEOREM

Piotr Niemiec

Abstract. The aim of the paper is to prove that if L is a linear subspace of the space $\mathcal{C}(K)$ of all real-valued continuous functions defined on a nonempty compact Hausdorff space K such that $\min(|f|,1) \in L$ whenever $f \in L$, then for any nonzero $g \in \bar{L}$ (where $\bar{L}$ denotes the uniform closure of L in $\mathcal{C}(K)$) and for any sequence $(b_n)_{n=1}^{\infty}$ of positive numbers satisfying the relation $\sum_{n=1}^{\infty} b_n = \|g\|$ there exists a sequence $(f_n)_{n=1}^{\infty}$ of elements of L such that $\|f_n\| = b_n$ for each $n \geqslant 1$, $g = \sum_{n=1}^{\infty} f_n$ and $|g| = \sum_{n=1}^{\infty} |f_n|$. Also the formula for $\bar{L}$ is given.

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1. INTRODUCTION

In the 19th century Weierstrass proved that every continuous function defined on the interval $[0,1]$ can be uniformly approximated by polynomials. Later Stone [10,11] generalized that result as follows: if K is a compact Hausdorff space and $\mathcal{A}$ is a subalgebra of $\mathcal{C}(K)$ which contains all constant functions and separates points of K (i.e. if for any two distinct points a and b of K there exists a function $f \in \mathcal{A}$ such that $f(a) \neq f(b)$), then $\mathcal{A}$ is dense in $\mathcal{C}(K)$ in the topology of uniform convergence. This fact is known as *the Stone-Weierstrass theorem.* A simple proof of it is based on the following property:

($\star$) *If* $\max(f,g), \min(f,g) \in \mathcal{F}$ *for any elements* f *and* g *of a subfamily* $\mathcal{F}$ *of* $\mathcal{C}(K)$ *and if* $g\colon K \to \mathbb{R}$ *is such a continuous function that for any* $x, y \in K$ *there exists* $f \in \mathcal{F}$ *satisfying the conditions* $f(x) = g(x)$ *and* $f(y) = g(y)$, *then* g *belongs to the uniform closure of the family* $\mathcal{F}$.

The Stone-Weierstrass theorem has many generalizations. For example, Glimm [5] proved its counterpart for arbitrary (noncommutative) $\mathcal{C}^*$-algebras, Bishop [2] generalized it to anti-symmetric algebras, Hofmann [6] formulated the categorical version of it and Garrido and Montalvo [4] generalized it to completely regular spaces. We strengthen the Stone-Weierstrass theorem for special linear subspaces of $\mathcal{C}(K)$, as described in the Abstract. Our result is, in a sense, in the spirit of classical Bernstein's lethargy theorem [1] (for a generalization see e.g. [7]) because it gives some information on the behavior of the sequence which approximates the given element of the space.

2. MAIN RESULT

In this paper K is a nonempty compact Hausdorff space, $\mathcal{C}(K)$ denotes the real algebra of all continuous real-valued functions on K equipped with the topology of uniform convergence and with the supremum norm $\|\cdot\|$, and L is a linear subspace of $\mathcal{C}(K)$ satisfying the condition:

$$\forall f \in L\colon\ \ \min(|f|, 1) \in L. \tag{2.1}$$

The space L has the following properties, the proofs of which are quite simple. Whenever $f, f_1, \ldots, f_n \in L$ and $t > 0$, then:

(L1) $|f| \in L$,
(L2) $\max(f_1, \ldots, f_n), \min(f_1, \ldots, f_n) \in L$,
(L3) $f_+, f_- \in L$, where $f_+ = \max(f, 0)$ and $f_- = \max(-f, 0)$,
(L4) $\min(f, t), \max(f, -t) \in L$.

The main theorem is preceded by the following lemma.

Lemma 2.1. *Let $h \in \bar{L}$ be a (nonzero) nonnegative function and let $t \in [0, \|h\|)$. Then there exists $f \in L$ such that $\|f\| = t$, $\|h - f\| = \|h\| - t$ and $0 \leqslant f \leqslant h$.*

Proof. Let $\varepsilon = \frac{1}{3}(\|h\| - t)\ > 0$. There exists $f_1 \in L$ such that $\|h - f_1\| \leqslant \varepsilon$. Let $f_2 = \max(f_1, 0)$. Thanks to (L3), $f_2 \in L$. Since $h \geqslant 0$ and the function $\mathbb{R} \ni x \mapsto \max(x, 0) \in \mathbb{R}$ is nonexpansive, i.e. $|\max(x, 0) - \max(y, 0)| \leqslant |x - y|$ for any $x, y \in \mathbb{R}$, we conclude that

$$\|h - f_2\| \leqslant \varepsilon. \tag{2.2}$$

Further, let $f_3 = f_2 - 2\min(f_2, \varepsilon)$. By (L4), $f_3 \in L$. Moreover, $f_3 \leqslant h$. Indeed, thanks to (2.2), $f_2(x) \leqslant h(x) + \varepsilon$. So, if $f_2(x) \geqslant \varepsilon$, then $f_3(x) = f_2(x) - 2\varepsilon \leqslant h(x)$. On the other hand, if $f_2(x) \leqslant \varepsilon$, then $f_3(x) = -f_2(x) \leqslant 0 \leqslant h(x)$.

Now let $f_4 = \max(f_3, 0)$. Then $f_4 \in L$ and $0 \leqslant f_4 \leqslant h$. Finally put $f = \min(f_4, t)\ \in L$. We easily see that $0 \leqslant f \leqslant h$ and $\|f\| \leqslant t$. It is enough to check that $\|h - f\| \leqslant \|h\| - t$. Let $x \in K$. If $h(x) \leqslant \|h\| - t$, then clearly $h(x) - f(x) \leqslant \|h\| - t$. So we may assume that $h(x) \geqslant \|h\| - t = 3\varepsilon$. Then, by (2.2), $f_2(x) \geqslant h(x) - \varepsilon \geqslant 2\varepsilon$ and hence $f_4(x) = f_3(x) = f_2(x) - 2\varepsilon \geqslant 0$. Now if $f_4(x) \geqslant t$, then $f(x) = t$ and $h(x) - f(x) \leqslant \|h\| - t$. On the other hand, if $f_4(x) \leqslant t$, then $f(x) = f_2(x) - 2\varepsilon$ and finally $h(x) - f(x) = h(x) - f_2(x) + 2\varepsilon \leqslant \|h - f_2\| + 2\varepsilon \leqslant 3\varepsilon = \|h\| - t$. □

Theorem 2.2. *Let $g \in \bar{L}$ be a nonzero function. Let $(b_n)_{n=1}^{\infty}$ be a sequence of positive numbers such that*

$$\sum_{n=1}^{\infty} b_n = \|g\|. \tag{2.3}$$

Then there exists a sequence $(f_n)_{n=1}^{\infty}$ *of elements of* L *such that* $\|f_n\| = b_n$ *for any* $n \geqslant 1$ *and* $g = \sum_{n=1}^{\infty} f_n$. *What is more,* $\{g \geqslant 0\} = \bigcap_{n=1}^{\infty}\{f_n \geqslant 0\}$ *and* $\{g \leqslant 0\} = \bigcap_{n=1}^{\infty}\{f_n \leqslant 0\}$. *In particular,* $|g| = \sum_{n=1}^{\infty} |f_n|$.

Proof. First assume that $g \geqslant 0$. Since $b_n > 0$ for each $n \geqslant 1$ and thanks to (2.3), $\sum_{k=1}^{n} b_k < \|g\|$. An easy use of the induction argument ensures us that, thanks to Lemma 2.1, there exists a sequence $(f_n)_{n=1}^{\infty}$ of elements of L such that $f_n \geqslant 0$, $\sum_{k=1}^{n} f_k \leqslant g$, $\|f_n\| = b_n$ and

$$\Big\|g - \sum_{k=1}^{n} f_k\Big\| = \|g\| - \sum_{k=1}^{n} b_k \tag{2.4}$$

for every $n \geqslant 1$. Indeed, if $f_1, \ldots, f_{n-1}$ are found, apply Lemma 2.1 for $h = g - \sum_{k=1}^{n-1} f_{n-1}$ and $t = b_n$ to obtain the function f_n.

We conclude from (2.3) and (2.4) that the series $\sum_{n=1}^{\infty} f_n$ is uniformly convergent to g and therefore in case of nonnegative g the proof is finished.

Now let g be an arbitrary element of $\bar{L}$. By (L3) and the continuity of the operators $f \mapsto f_+$ and $f \mapsto f_-$, $g_+, g_- \in \bar{L}$. Observe that

$$g_+ \cdot g_- \equiv 0 \qquad \text{and} \qquad \|g\| = \max(\|g_+\|, \|g_-\|). \tag{2.5}$$

The second of the above connections, combined with (2.3), implies that there exist two sequences $(b_n^+)_{n=1}^{\infty}$ and $(b_n^-)_{n=1}^{\infty}$ of positive numbers such that

$$\sum_{n=1}^{\infty} b_n^+ = \|g_+\|, \quad \sum_{n=1}^{\infty} b_n^- = \|g_-\| \quad \text{and} \quad \max(b_n^+, b_n^-) = b_n. \tag{2.6}$$

Now we may apply the first part of the proof for g_+ and g_- to obtain two corresponding sequences $(f_n^+)_{n=1}^{\infty}$ and $(f_n^-)_{n=1}^{\infty}$ of nonnegative elements of L satisfying the equalities $g_\pm = \sum_{n=1}^{\infty} f_n^\pm$ and $\|f_n^\pm\| = b_n^\pm$ $(n \geqslant 1)$. To end the construction, put $f_n = f_n^+ - f_n^-$ and observe that:

(i) $f_n^+ \cdot f_n^- \equiv 0$ (thanks to (2.5) and the inequality $0 \leqslant f_n^\pm \leqslant g_\pm$), and hence $\|f_n\| = \max(\|f_n^+\|, \|f_n^-\|) = b_n$ (by (2.6)),

(ii) if $g(x) \geqslant 0$ [$g(x) \leqslant 0$], then $g_-(x) = 0$ [$g_+(x) = 0$], so $f_n^-(x) = 0$ [$f_n^+(x) = 0$] for each $n \geqslant 1$ and therefore $f_n(x) \geqslant 0$ [$f_n(x) \leqslant 0$]. □

3. SOME APPLICATIONS

Theorem 2.2 cannot be applied for L being the space of all real-valued polynomials on the interval $[0, 1]$, because this space does not satisfy the crucial condition (2.1). However, it is well known that if (K, d) is a compact metric space, then the space $\mathrm{Lip}(K)$ consisting of all real-valued *Lipschitz* functions on K ($g\colon K \to \mathbb{R}$ belongs to $\mathrm{Lip}(K)$ if there exists a constant $M \in [0, \infty)$ such that $|g(x) - g(y)| \leqslant Md(x, y)$ for every $x, y \in K$) is a subalgebra of $\mathcal{C}(K)$ which separates points of K. What is more, if $f \in \mathrm{Lip}(K)$, then $\min(|f|, 1) \in \mathrm{Lip}(K)$.

So, a special case of Theorem 2.2 is the following statement.

Proposition 3.1. *If* (K,d) *is a nonempty compact metric space and* $g \in \mathcal{C}(K)$*, then there exists a sequence* $(f_n)_{n=1}^{\infty}$ *of real-valued Lipschitz functions such that* $\|g\| = \sum_{n=1}^{\infty} \|f_n\|$, $g = \sum_{n=1}^{\infty} f_n$ *and* $|g| = \sum_{n=1}^{\infty} |f_n|$.

Proposition 3.1 is applied in [9] to establish an important property of the function linear space $\mathrm{CFL}(\mathbb{U})$, whose elements are the uniform limits of linear combinations of maps of the form $\mathbb{U} \ni x \mapsto d(x,y) - d(x,z) \in \mathbb{R}$ with $y,z \in \mathbb{U}$, generated by the Urysohn universal metric space $(\mathbb{U},d)$ ($\mathbb{U}$ is uniquely determined by its diameter and the following properties: it is separable and complete, every separable metric space of diameter no greater than $\operatorname{diam}\mathbb{U}$ is isometrically embeddable in $\mathbb{U}$, and every isometric map between finite subsets of $\mathbb{U}$ is extendable to an isometry of $\mathbb{U}$), namely: if K is a compact subset of $\mathbb{U}$ and $f\colon K \to \mathbb{R}$ is a continuous function, then there exists an extension $F \in \mathrm{CFL}(\mathbb{U})$ of f such that $\|F\| = \|f\|$. This result enables us to build an example of an adjoint linear isomoprhism between dual Banach spaces which is an isometry on the weakly-* dense subspace but not on the whole domain.

In case when L is a subspace of $\mathcal{C}(K)$, the closure of L can be nicely described. To do that, we put the definition.

Definition 3.2. *The null set of the space* L *is the set* $\mathcal{N}(L) = \{x \in K\colon\ f(x) = 0 \text{ for each } f \in L\}$. *The equivalence relation* $\mathcal{R}(L)$ *on* K *induced by* L *is defined by the formula:*

$$(x,y) \in \mathcal{R}(L) \iff \forall f \in L\colon\ f(x) = f(y) \qquad (x,y \in K).$$

The algebra generated by L *is the algebra*

$$\mathcal{A}(\mathcal{N}(L),\mathcal{R}(L)) = \{g \in \mathcal{C}(K)|\ g\big|_{\mathcal{N}(L)} \equiv 0, \quad \forall (x,y) \in \mathcal{R}(L)\colon\ g(x) = g(y)\}.$$

The sets $\mathcal{N}(L)$ *and* $\mathcal{R}(L)$ *are closed subsets of* K *and* $K \times K$*, respectively, and the algebra* $\mathcal{A}(\mathcal{N}(L),\mathcal{R}(L))$ *is a closed subalgebra of* $\mathcal{C}(K)$*, possibly with no unit.*

The following result, which has entered folklore (cf. [3]), explains the terminology. For the reader's convenience, we give a short proof.

Proposition 3.3. *The closure of the space* L (*satisfying the condition* (2.1)) *in the space* $\mathcal{C}(K)$ *coincides with* $\mathcal{A}(\mathcal{N}(L),\mathcal{R}(L))$.

Proof. Clearly $\bar{L} \subset \mathcal{A}(\mathcal{N}(L),\mathcal{R}(L))$. To see the inverse inclusion, take $g \in \mathcal{A}(\mathcal{N}(L),\mathcal{R}(L))$. Observe that:

$$\forall x,y \in K\ \exists f \in L\colon\ f(x) = g(x),\ f(y) = g(y). \tag{3.1}$$

Indeed, the following five conditions are possible:

(1°) $x,y \in \mathcal{N}(L)$: take $f = 0$.
(2°) $x \in \mathcal{N}(L)$ and $y \notin \mathcal{N}(L)$ (or conversely): there exists $f_0 \in L$ such that $f_0(y) \neq 0$. Now it is enough to put $f = \frac{g(y)}{f_0(y)} f_0$.
(3°) $x,y \notin \mathcal{N}(L)$ and $(x,y) \in \mathcal{R}(L)$: do the same as in (2°).

(4°) $x, y \notin \mathcal{N}(L)$, $(x,y) \notin \mathcal{R}(L)$ and $g(x) = g(y)$: there exist $f_1, f_2 \in L$ such that $f_1(x) \cdot f_2(y) \neq 0$. Let $f_0 = |f_1| + |f_2|$. By (L1), $f_0 \in L$. Let $m = \min(f_0(x), f_0(y)) > 0$ and finally put $f = \frac{g(x)}{m} \min(f_0, m) \in L$.

(5°) $x, y \notin \mathcal{N}(L)$, $(x,y) \notin \mathcal{R}(L)$ and $g(x) \neq g(y)$: there exists $f_1 \in L$ such that $f_1(x) \neq f_1(y)$. By the proof of (4°), there is $f_2 \in L$ such that $f_2(x) = f_2(y) = g(x) - \frac{g(x)-g(y)}{f_1(x)-f_1(y)} f_1(x)$. Now it is easy to check that $f(x) = g(x)$ and $f(y) = g(y)$ for $f = \frac{g(x)-g(y)}{f_1(x)-f_1(y)} f_1 + f_2 \in L$.

Having (3.1), it suffices to apply the property ($\star$). □

Now we shall give some illustrative examples dealing with the subject.

Examples 3.4. *In everywhere below, Ω is a nonempty compact Hausdorff space and each of the spaces L appearing below consists of continuous real-valued functions on Ω and satisfies* (2.1)*.*

A. *Suppose Ω is totally disconnected. The space L of all functions with finite images is dense in $\mathcal{C}(\Omega)$.*
B. *Let U be an open nonempty subset of Ω and let L constist of all functions whose support is contained in U; that is, $f \in L$ iff* $\operatorname{supp} f := \overline{f^{-1}(\mathbb{R} \setminus \{0\})} \subset U$. *Then $\bar{L}$ constists of all functions vanishing on $\Omega \setminus U$.*
C. *Suppose Ω is metrizable and d is a metric on Ω which induces the topology of Ω. For a fixed $p > 0$ let L be the space of all functions satisfying the Hölder condition with exponent p.* (*L may not be dense in $\mathcal{C}(\Omega)$ for $p > 1$. It may even constists only of constant functions, as it is in case of $\Omega = [0,1]$ with the natural metric.*)
D. *Let Ω and d be as in the previous example and let L be the space of all the so-called little Lipschitz functions on Ω (cf.* [12, *Chapter 3*]*); that is, $f \in L$ iff for every $\varepsilon > 0$ there is $\delta > 0$ such that $|f(x) - f(y)| \leqslant \varepsilon d(x,y)$ whenever $d(x,y) \leqslant \delta$.* (*L may consists only of constant functions. See* [12] *for utility of this space.*)
E. *Let $\Omega = [a,b] \subset \mathbb{R}$ and let L be the space of all piecewise affine functions. Then L is dense in $\mathcal{C}(\Omega)$ and it is* ***not*** *an algebra.*
F. *Let A be a countable subset of Ω and let L constist of all f such that $\sum_{a \in A} |f(a)| < +\infty$. L is a proper* (*nonclosed*) *ideal in $\mathcal{C}(\Omega)$. One may show that L is dense in $\mathcal{C}(\Omega)$ provided the topology of A is discrete.* (*Indeed, in the latter case L separates points of Ω and does not vanish at every point.*)

We end the paper with the note that condition (2.1) is crucial in the classical theory of the *Daniell-Stone integral* (cf. [8]) and therefore we believe our result may find application there.

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Piotr Niemiec
piotr.niemiec@uj.edu.pl

Jagiellonian University
Institute of Mathematics
ul. Łojasiewicza 6, 30-348 Krakow, Poland

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