

**NOTE ON THE STABILITY
OF FIRST ORDER
LINEAR DIFFERENTIAL EQUATIONS**

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Abstract. In this paper, we will prove the generalized Hyers-Ulam stability of the linear differential equation of the form $y'(x)+f(x)y(x)+g(x)=0$ under some additional conditions.

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1. INTRODUCTION

The study of the stability functional equations is strongly related to Ulam's question concerning the stability of group homomorphisms. We mention that the concept of stability for a functional equation appears when we replace the functional equation by an inequality which acts as a perturbation of the equation. Thus the stability question for functional equations shows "how the solutions of the inequality differ from those of the given functional equation." D.H. Hyers [3] excellently answered the question of Ulam and proved the following result:

Theorem 1.1 (Hyers, [3]). *Let E and E' be two Banach spaces and $f : E \rightarrow E'$ a given function such that there exists $\delta \geq 0$ such that*

$$\|f(x+y) - f(x) - f(y)\| \leq \delta, \quad \forall x, y \in X. \quad (1.1)$$

Then the limit $L(x) = \lim_{n \rightarrow \infty} \frac{f(2^n x)}{2^n}$ exists for all $x \in E$, L is an additive function and the inequality

$$\|f(x) - L(x)\| < \delta \quad (1.2)$$

is true for all $x \in E$. Moreover, $L(x)$ is the only additive function which satisfies the inequality (1.2).

Since Hyers' result, a great number of papers on the subject have been published, extending and generalizing the Ulam's problem and the Hyers' theorem in various directions (see [3, 9, 10]).

In [9] V. Radu proposed a new method for obtaining the existence of exact solutions and error estimations, based on the fixed point alternative and this theorem is:

Theorem 1.2 (The fixed point alternative). *Suppose we are given a complete generalized metric space (Ω, d) and a strictly contractive mapping $T : \Omega \rightarrow \Omega$ with the Lipschitz constant a . Then, for each given element $x \in \Omega$, either*

$$d(T^n x, T^{n+1} x) = \infty, \quad \forall n \geq 0,$$

or there exists a natural number n_0 such that:

- (i) $d(T^n x, T^{n+1} x) < \infty$ for all $n \geq n_0$.
- (ii) The sequence $(T^n x)_{n \geq 0}$ is convergent to a fixed point y^* of T .
- (iii) y^* is the unique fixed point of T in the set $\Delta = \{y \in \Omega \mid d(T^{n_0} x, y) < \infty\}$.
- (iv) $d(y, y^*) \leq \frac{1}{1-a} d(y, Ty)$ for all $y \in \Delta$.

Let a_0, a_1, \dots, a_{n-1} be real numbers and let I be an interval. For $y \in C^n(I, \mathbb{R})$, $\varepsilon > 0$ and $\varphi \in C(I, \mathbb{R}_+)$ we consider the following equation:

$$y^{(n)}(t) = \sum_{k=0}^{n-1} a_k y^{(k)}(t), \quad t \in I \quad (1.3)$$

and the following inequations

$$\left| y^{(n)}(t) - \sum_{k=0}^{n-1} a_k y^{(k)}(t) \right| \leq \varepsilon, \quad t \in I \quad (1.4)$$

and

$$\left| y^{(n)}(t) - \sum_{k=0}^{n-1} a_k y^{(k)}(t) \right| \leq \varphi(t), \quad t \in I. \quad (1.5)$$

Definition 1.3. The equation (1.3) is Hyers-Ulam stable if there exists a real number $c > 0$ such that for each $\varepsilon > 0$ and for each solution $s \in C^{(n)}(I, \mathbb{R})$ of (1.4) there exists a solution $y \in C^{(n)}(I, \mathbb{R})$ of (1.3) with

$$|s(t) - y(t)| \leq c \cdot \varepsilon, \quad \forall t \in I.$$

Definition 1.4. The equation (1.3) is Hyers-Ulam-Rassias stable, with respect to φ , if there exists a real number $c_\varphi > 0$ such that for each solution $s \in C^{(n)}(I, \mathbb{R})$ of (1.5) there exists a solution $y \in C^{(n)}(I, \mathbb{R})$ of (1.3) with

$$|s(t) - y(t)| \leq c_\varphi \cdot \varphi(t), \quad \forall t \in I.$$

Alsina and Ger were the first authors who investigated the Hyers-Ulam stability of differential equations. In 1998, they proved in [1] the stability of differential equation

$$y'(t) = y(t). \tag{1.6}$$

Following the same approach as in [1], Miura [8] proved the Hyers-Ulam stability of differential equation

$$y'(t) = \lambda y(t). \tag{1.7}$$

S.M. Jung [4–7] applied the fixed point method for proving the Hyers-Ulam-Rassias stability of a Volterra integral equation of the second kind and for differential equations of first order. Using the same technique we prove the Hyers-Ulam-Rassias stability and Hyers-Ulam stability of differential equation

$$y'(x) + f(x)y(x) + g(x) = 0 \tag{1.8}$$

under some conditions, others than the conditions from [4].

2. MAIN RESULTS

In this paper, by using the idea of Cădariu and Radu [2], we will prove the Hyers-Ulam-Rassias stability for the equation (1.8) on the intervals $I = [a, b)$, where $-\infty < a < b \leq \infty$.

Theorem 2.1. *Let $f, g : I \rightarrow \mathbb{R}$ be continuous functions and let for a positive constant M , $|f(x)| \geq M$ for all $x \in I$. Assume that $\psi : I \rightarrow [0, \infty)$ is an integrable function with the property that there exists $P \in (0, 1)$ such that*

$$\int_a^x |f(t)|\psi(t)dt \leq P\psi(x) \tag{2.1}$$

for all $x \in I$. If a continuously differentiable function $y : I \rightarrow \mathbb{R}$ verifies the relation:

$$|y'(x) + f(x)y(x) + g(x)| \leq \psi(x) \tag{2.2}$$

for all $x \in I$, then there exists a unique solution $S : I \rightarrow \mathbb{R}$ of the equation (1.8) which verifies the following relations:

$$|y(x) - S(x)| \leq \frac{P}{M - MP}\psi(x) \tag{2.3}$$

for all $x \in I$ and $S(a) = y(a)$.

Proof. Let us consider the set $\Omega = \{h : I \rightarrow \mathbb{R} \mid h \text{ is continuous and } h(a) = y(a)\}$ and the generalized metric $d(h_1, h_2)$ defined on Ω as

$$d(h_1, h_2) = d_\psi(h_1, h_2) = \inf \{k > 0 \mid |h_1(x) - h_2(x)| \leq k\psi(x), \forall x \in I\}.$$

Then (Ω, d) is a generalized complete metric space (see [4]). We define the operator $T : \Omega \rightarrow \Omega$,

$$Th(x) = y(a) - \int_a^x (f(t)h(t) + g(t))dt \quad x \in I,$$

for all $h \in \Omega$. Indeed Th is a continuously differentiable function on I , since f and g are continuous function and $Th(a) = y(a)$.

Now, let $h_1, h_2 \in \Omega$. Then we have

$$\begin{aligned} |Th_1(x) - Th_2(x)| &= \left| \int_a^x f(t)(h_1(t) - h_2(t)) dt \right| \leq \int_a^x |f(t)| |h_1(t) - h_2(t)| dt \leq \\ &\leq d(h_1, h_2) \int_a^x |f(t)| \psi(t) dt \leq P\psi(x)d(h_1, h_2) \end{aligned}$$

for all $x \in I$. Therefore,

$$d(Th_1, Th_2) \leq Pd(h_1, h_2), \quad (2.4)$$

thus the operator T is a contraction with the constant P .

Now integrating the both sides of the relation (2.2) on $[a, x]$ we obtain

$$\left| y(x) - y(a) + \int_a^x (f(t)y(t) + g(t)) dt \right| \leq \frac{P}{M}\psi(x) \quad (2.5)$$

for all $x \in I$, which means $d(y, Ty) \leq \frac{P}{M} < \infty$. By the fixed point alternative there exists an element $S = \lim_{n \rightarrow \infty} T^n y$ and S is unique fixed point of T in the set $\Delta = \{h \in \Omega \mid d(T^{n_0}y, h) < \infty\}$. It may be proved that

$$\Delta = \{h \in \Omega \mid d(y, h) < \infty\}.$$

Therefore the set Δ is independent of n_0 . To prove that the function S is a solution to the equation (1.8), we derive with respect to x the both sides of the relation

$$S(x) = TS(x), \quad x \in I. \quad (2.6)$$

Thus

$$S'(x) = -f(x)S(x) - g(x) \quad (2.7)$$

for all $x \in I$ which implies that the function S is a solution to the equation (1.8) and verifies the relation $S(a) = y(a)$.

Applying again the fixed point alternative we obtain

$$d(h, S) \leq \frac{1}{1-P}d(h, Th) \quad \text{for all } h \in \Delta.$$

Since $y \in \Delta$, we have

$$d(y, S) \leq \frac{1}{1 - P}d(y, Ty) \leq \frac{P}{M(1 - P)},$$

whence

$$|y(x) - S(x)| \leq \frac{P}{M - MP}\psi(x)$$

for all $x \in I$. This inequality proves the relation (2.3). □

In the same manner it can be proved the following theorem of the Hyers-Ulam-Rassias stability of the equation (1.8) on the interval $J = (b, a]$, where $-\infty \leq b < a < \infty$.

Theorem 2.2. *Let $f, g : J \rightarrow \mathbb{R}$ be continuous functions and let for some positive constant M , $|f(x)| \geq M$ for all $x \in J$. Assume that $\psi : J \rightarrow [0, \infty)$ is an integrable function with the property that there exists $P \in (0, 1)$ such that*

$$\int_x^a |f(t)|\psi(t)dt \leq P\psi(x) \tag{2.8}$$

for all $x \in J$. If a continuously differentiable function $y : J \rightarrow \mathbb{R}$ verifies the relation:

$$|y'(x) + f(x)y(x) + g(x)| \leq \psi(x) \tag{2.9}$$

for all $x \in J$, then there exists a unique solution $S : J \rightarrow \mathbb{R}$ of the equation (1.8) which verifies the following relations:

$$|y(x) - S(x)| \leq \frac{P}{M - MP}\psi(x) \tag{2.10}$$

for all $x \in J$ and $S(a) = y(a)$.

The Hyers-Ulam-Rassias stability equation (1.8) on \mathbb{R} will be proved by Theorem 2.1 and Theorem 2.2.

Corollary 2.3. *Let $f, g : \mathbb{R} \rightarrow \mathbb{R}$ be continuous functions and let for some positive constant M , $|f(x)| \geq M$ for all $x \in \mathbb{R}$. Assume that $\psi : \mathbb{R} \rightarrow [0, \infty)$ is an integrable function with the property that there exists $P \in (0, 1)$ such that*

$$\left| \int_0^x |f(t)|\psi(t)dt \right| \leq P\psi(x) \tag{2.11}$$

for all $x \in \mathbb{R}$. If a continuously differentiable function $y : \mathbb{R} \rightarrow \mathbb{R}$ verifies the relation:

$$|y'(x) + f(x)y(x) + g(x)| \leq \psi(x) \tag{2.12}$$

for all $x \in \mathbb{R}$, then there exists a unique solution $S : \mathbb{R} \rightarrow \mathbb{R}$ of equation (1.8) which verifies the following relations:

$$|y(x) - S(x)| \leq \frac{P}{M - MP} \psi(x) \quad (2.13)$$

for all $x \in \mathbb{R}$ and $S(0) = y(0)$.

Proof. By the relation (2.11) we have

$$\int_0^x |f(t)|\psi(t)dt \leq P\psi(x) \quad (2.14)$$

for all $x \geq 0$. Applying Theorem 2.1, there exists a solution of equation (1.8), $S_1 : [0, \infty) \rightarrow \mathbb{R}$ which verifies the relations (2.3) and $S_1(0) = y(0)$.

From (2.11) we also obtain

$$\int_x^0 |f(t)|\psi(t)dt \leq P\psi(x) \quad (2.15)$$

for all $x \leq 0$. Applying Theorem 2.2, there exists a solution of equation (1.8), $S_2 : (-\infty, 0] \rightarrow \mathbb{R}$ which verifies (2.10) and $S_2(0) = y(0)$. It is easy to check that the function

$$S(x) = \begin{cases} S_1(x), & x \geq 0, \\ S_2(x), & x < 0, \end{cases} \quad (2.16)$$

is a continuously differentiable function on \mathbb{R} , a solution of equation (1.8) on \mathbb{R} and it verifies relation (2.13). \square

Using Theorem 2.1 it can be shown the Hyers-Ulam stability for the equation (1.8) on $I = [a, b]$, where $-\infty < a < b \leq \infty$.

Corollary 2.4. *Let $\varepsilon, M > 0$ and let $f : I \rightarrow [M, \infty)$ and $g : I \rightarrow \mathbb{R}$ be continuous. If a continuously differentiable function $y : I \rightarrow \mathbb{R}$ verifies the relation*

$$|y'(x) + f(x)y(x) + g(x)| \leq \varepsilon \quad (2.17)$$

for all $x \in I$, then there exists a unique solution $S : I \rightarrow \mathbb{R}$ of equation (1.8) which verifies the relations:

$$|y(x) - S(x)| \leq \frac{\varepsilon}{M(2q - 1)} \quad (2.18)$$

for all $x \in I$, where $q \in (\frac{1}{2}, 1)$ and $S(a) = y(a)$.

Proof. Let $q \in (\frac{1}{2}, 1)$. Multiplying relation (2.17) by $e^{q \int_a^x f(t)dt}$, and denoting

$$z(x) := y(x)e^{q \int_a^x f(t)dt}, \quad x \in I \quad (2.19)$$

we have

$$\left| z'(x) + (1 - q) f(x)z(x) + g(x)e^q \int_a^x f(t)dt \right| \leq \varepsilon \cdot e^q \int_a^x f(t)dt \tag{2.20}$$

for all $x \in I$. Then the function $F(x) = (1 - q) f(x)$, where $x \in I$, is continuous on I and satisfies the relation $|F(x)| > (1 - q) M$ for all $x \in I$.

Let $\psi(x) = \varepsilon \cdot e^q \int_a^x f(t)dt$, when $x \in I$. We see that

$$\int_a^x |F(t)| \psi(t)dt = (1 - q) \varepsilon \int_a^x f(t)e^q \int_a^t f(u)du dt \leq \frac{1 - q}{q} \psi(x) \tag{2.21}$$

for all $x \in I$, thus the function $\psi : I \rightarrow [0, \infty)$ verifies relation (2.1) with $P = \frac{1 - q}{q} \in (0, 1)$.

By Theorem 2.1, there exists $s \in C^1(I, \mathbb{R})$, which is a unique solution for the equation

$$z'(x) + (1 - q) f(x)z(x) + g(x)e^q \int_a^x f(t)dt = 0 \tag{2.22}$$

and verifies the relations

$$|z(x) - s(x)| \leq \frac{1}{M(2q - 1)} \cdot \varepsilon \cdot e^q \int_a^x f(t)dt \tag{2.23}$$

for all $x \in I$ and $s(a) = z(a)$.

Then the function $S(x) = s(x)e^{-q \int_a^x f(t)dt}$ is a solution of equation (1.8) and verifies relation (2.18). \square

Equation (1.8) is not Hyers-Ulam stable on the intervals $J = (-\infty, a]$ in general, as we can see in the following example.

Example 2.5. Let us consider equation (1.8) where $f(x) = x^2$ and $g(x) = 0$. The solution of this equation $S : J \rightarrow \mathbb{R}$ which verifies the condition $S(a) = p$ is

$$S(x) = p \cdot e^{\frac{a^3 - x^3}{3}}. \tag{2.24}$$

A continuously differentiable function $y : J \rightarrow \mathbb{R}$ which verifies inequality (2.17) is

$$y(x) = p \cdot e^{\frac{a^3 - x^3}{3}} + \varepsilon \cdot e^{-\frac{x^3}{3}} \int_a^x e^{\frac{t^3}{3}} dt. \tag{2.25}$$

Considering equation (1.8) being Hyers-Ulam stable, there exists $k > 0$ such that

$$|y(x) - S(x)| \leq \varepsilon \cdot k \tag{2.26}$$

for all $x \in J$. By substitution, we have

$$\left| \int_a^x e^{\frac{t^3}{3}} dt \right| \leq k e^{\frac{x^3}{3}} \tag{2.27}$$

for all $x \in J$. Now letting $x \rightarrow -\infty$ it generates a contradiction. So equation (1.8) is not Hyers-Ulam stable.

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