GLOBAL OFFENSIVE k-ALLIANCE IN BIPARTITE GRAPHS

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Abstract. Let $k \geq 0$ be an integer. A set S of vertices of a graph $G = (V(G), E(G))$ is called a global offensive k-alliance if $|N(v) \cap S| \geq |N(v) - S| + k$ for every $v \in V(G) - S$, where $0 \leq k \leq \Delta$ and Δ is the maximum degree of G. The global offensive k-alliance number $\gamma_o^k(G)$ is the minimum cardinality of a global offensive k-alliance in G. We show that for every bipartite graph G and every integer $k \geq 2$, $\gamma_o^k(G) \leq \frac{n(G)+|L_k(G)|}{2}$, where $L_k(G)$ is the set of vertices of degree at most $k-1$. Moreover, extremal trees attaining this upper bound are characterized.

Keywords: global offensive k-alliance number, bipartite graphs, trees.

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1. INTRODUCTION

We begin with some terminology. For a vertex v of a graph $G = (V, E) = (V(G), E(G))$, the *open neighborhood* of a vertex $v \in V$ is $N(v) = \{u \in V \mid uv \in E\}$ and the *closed neighborhood* is $N[v] = N(v) \cup \{v\}$. The *degree* of v, denoted by $\deg_G(v)$, is $|N(v)|$. By $n(G)$ and $\Delta(G) = \Delta$ we denote the order and the maximum degree of the graph G, respectively. Specifically, for a vertex v in a rooted tree T, we denote by $C(v)$ and $D(v)$ the set of children and descendants, respectively, of v, and we define $D[v] = D(v) \cup \{v\}$. The *maximal subtree* at v is the subtree of T induced by $D[v]$, and is denoted by T_v.

In [3] Kristiansen, Hedetniemi, and Hedetniemi introduced several types of alliances in graphs, including defensive and offensive alliances. We are interested in a generalization of offensive alliances, namely global offensive k-alliances given by Shafique and Dutton [4, 5]. Let $k \geq 0$ be an integer. A set S of vertices of a graph G is called a *global offensive k-alliance* if $|N(v) \cap S| \geq |N(v) - S| + k$ for every $v \in V(G) - S$ for $0 \leq k \leq \Delta$. The *global offensive k-alliance number* $\gamma_o^k(G)$ is the minimum cardinality of a global offensive k-alliance in G. If S is a global offensive k-alliance of G and $|S| = \gamma_o^k(G)$, then we say that S is a $\gamma_o^k(G)$-set. A global offensive

1-alliance is a global offensive alliance and a global offensive 2-alliance is a global strong offensive alliance.

In this paper, we show that for every bipartite graph G and every integer $k \geq 1$, $\gamma_o^k(G) \leq \frac{n(G)+|L_k(G)|}{2}$, where $L_k(G) = \{x \in V(G) : \deg_G(x) \leq k-1\}$. Moreover, extremal trees attaining the upper bound are characterized for $k \geq 2$.

2. MAIN RESULTS

Theorem 2.1. *Let $k \geq 1$ be an integer. If G is a bipartite graph, then*

$$\gamma_o^k(G) \leq \frac{n(G)+|L_k(G)|}{2}.$$

Proof. Let G be a bipartite graph. Clearly, $L_k(G)$ is contained in every $\gamma_o^k(G)$-set. Let H be the graph obtained from G by removing $L_k(G)$. If H is empty, then the result is valid. Thus we assume now that $n(H) \geq 1$, and so H admits a bipartition A, B, where $A = \emptyset$ or $B = \emptyset$ is possible. Every vertex of A (resp., B) has at least k neighbors in $B \cup L_k(G)$ (resp., $A \cup L_k(G)$). It follows that each of $A \cup L_k(G)$ and $B \cup L_k(G)$ is a global offensive k-alliance of G and so

$$\gamma_o^k(G) \leq \min\{|A \cup L_k(G)|, |B \cup L_k(G)|\} \leq \\ \leq \frac{n(G)-|L_k(G)|}{2} + |L_k(G)| = \frac{n(G)+|L_k(G)|}{2}. \qquad \square$$

The case $k = 2$ in Theorem 2.1 leads to the next result.

Corollary 2.2 ([2]). *If G is a bipartite graph, then*

$$\gamma_o^2(G) \leq \frac{n(G)+|L_2(G)|}{2}.$$

For a positive integer k, a set of vertices D in a graph G is said to be a *k-dominating set* if each vertex of G not contained in D has at least k neighbors in D. The order of a smallest k-dominating set of G is called the *k-domination number*, and it is denoted by $\gamma_k(G)$. Clearly, if S is any $\gamma_o^k(G)$-set, then every vertex of $V(G) - S$ has at least k neighbors in S. Thus S is a k-dominating set of G, and hence $\gamma_k(G) \leq \gamma_o^k(G)$. Using this fact, Theorem 2.1 implies the following corollary.

Corollary 2.3 ([1]). *Let $k \geq 1$ be an integer. If G is a bipartite graph, then*

$$\gamma_k(G) \leq \frac{n(G)+|L_k(G)|}{2}.$$

In [1], Blidia, Chellali and Volkmann defined the following trees. For a positive integer p, a nontrivial tree T is called an *$\mathcal{N}_p$-tree* if T contains a vertex, say w, of degree at least $p-1$ and $\deg_T(x) \leq p-1$ for every vertex of $x \in V(T) - \{w\}$. We will call w the *special vertex* of T. An *$\mathcal{N}_p$-tree* with special vertex w is called *exact* if $\deg_T(w) = p-1$. The subdivided star $K_{1,p}$ $(p \geq 1)$ is an example of an *$\mathcal{N}_p$-tree*.

In order to characterize extremal trees achieving equality in Theorem 2.1 we define the family $\mathcal{F}_k$ of all trees T that can be obtained from a sequence $T_1, T_2, \ldots, T_p$ ($p \geq 1$) of trees, where T_1 is an exact $\mathcal{N}_k$-tree, $T = T_p$, and, if $p \geq 2$, T_{i+1} can be obtained recursively from T_i by one of the two operations listed below.

- Operation $\mathcal{O}_1$: Attach an $\mathcal{N}_k$-tree of special vertex w of degree at least $k+1$ by adding an edge from w to a vertex u of T_i of degree exactly $k-1$, and adding at most one new tree, all vertices of degree at most $k-1$ and join a vertex of degree at most $k-2$ with u by an edge.
- Operation $\mathcal{O}_2$: Attach an $\mathcal{N}_k$-tree of special vertex w of degree k or $k-1$ by adding an edge from w to a vertex u of T_i of degree exactly $k-1$, and adding t ($t \geq 0$) new trees, all vertices of degree at most $k-1$ and join a vertex of degree at most $k-2$ of each new tree with u by an edge.

We state a lemma.

Lemma 2.4. *If $T \in \mathcal{F}_k$, then $\gamma_o^k(T) = (n(T) + |L_k(T)|)/2$.*

Proof. Assume that $T \in \mathcal{F}_k$. Clearly, $\Delta(T) \geq k-1$ and T is obtained from a sequence $T_1, T_2, \ldots, T_p$ ($p \geq 1$) of trees, where T_1 is an exact $\mathcal{N}_k$-tree, $T = T_p$, and, if $p \geq 2$, T_{i+1} can be obtained recursively from T_i by one of the two operations defined above. We will use an induction on p. If $p = 1$, then T is an exact $\mathcal{N}_k$-tree where $\gamma_o^k(T) = |L_k(T)| = n(T)$ and so $\gamma_o^k(T) = (n(T) + |L_k(T)|)/2$.

Assume now that $p \geq 2$ and that the result holds for all trees $T \in \mathcal{F}_k$ that can be constructed from a sequence of length at most $p-1$, and let $T' = T_{p-1}$. By the inductive hypothesis on $T' \in \mathcal{F}_k$ we have $\gamma_o^k(T') = (n(T') + |L_k(T')|)/2$. Let T be a tree obtained from T' and S a $\gamma_o^k(T)$-set. We consider the following two cases.
Case 1. T is obtained from T' by using operation $\mathcal{O}_1$.

Let H be the $\mathcal{N}_k$-tree of special vertex w of degree at least $k+1$ added to T' and let Q be the new tree of maximum degree at most $k-1$ that can possibly be added to T'. Clearly $n(T) = n(T') + n(H) + n(Q)$ and $|L_k(T)| = |L_k(T')| + |V(H)| + |V(Q)| - 2$. Then S contains all vertices of Q, H except possibly w. If $w \in S$, then $u \notin S$ otherwise $S - \{w\}$ is a global offensive k-alliance of T, contradicting the minimality of S, but then $\{u\} \cup S - \{w\}$ is a $\gamma_o^k(T)$-set that contains u and not w. Now if $w \notin S$, then $u \in S$ otherwise since $k \leq \deg_T(u) \leq k+1$, $k \geq |N(u) \cap S| \geq |N(u) - S| + k \geq 1 + k$, which is impossible. Thus we may assume without loss of generality that $u \in S$ and $w \notin S$. Now let $S' = S \cap V(T')$. Since S is a $\gamma_o^k(T)$-set, every vertex of $z \in V(T') - S'$ satisfies $|N(z) \cap S'| \geq |N(v) - S'| + k$ and hence S' is a global offensive k-alliance of T', implying that $\gamma_o^k(T') \leq \gamma_o^k(T) - |V(H)| - |V(Q)| + 1$. Now since $\deg_{T'}(u) = k-1$, u is in every $\gamma_o^k(T')$-set, and such a set can be extended to a global offensive k-alliance of T by adding $(V(H) - \{w\}) \cup V(Q)$; and so $\gamma_o^k(T) \leq \gamma_o^k(T') + |V(H)| + |V(Q)| - 1$. It follows that $\gamma_o^k(T) = \gamma_o^k(T') + |V(H)| + |V(Q)| - 1$. Using induction on T', we obtain $\gamma_o^k(T) = (n(T) + |L_k(T)|)/2$.

Case 2. T is obtained from T' by using operation $\mathcal{O}_2$.

Let H be the $\mathcal{N}_k$-tree of special vertex w of degree $k-1$ or k added to T' and let $Q_1, Q_2, \ldots, Q_t$ be the $t \geq 0$ new trees that can possibly be added to T', each one of maximum degree at most $k-1$. Then

$$n(T) = n(T') + n(H) + \sum_{j=1}^{t} |V(Q_j)|,$$

and

$$|L_k(T)| = |L_k(T')| - 1 + |(V(H) - \{w\})| + \sum_{j=1}^{t} |V(Q_j)|.$$

Every $\gamma_o^k(T')$-set contains u and can be extended to a global offensive k-alliance of T by adding the set $V(H) - \{w\}$ and all the vertices of Q_j for every j, so

$$\gamma_o^k(T) \leq \gamma_o^k(T') + |V(H)| - 1 + \sum_{j=1}^{t} |V(Q_j)|.$$

On the other hand, $V(Q_j) \subset S$ for every j , $(V(H) - \{w\}) \subset S$ and S must contain one of w or u, otherwise S would not be a global offensive k-alliance of T since $|N(w) \cap S| = k < k+1 = |N(w) - S| + k$. Thus we may assume that $u \in S$, and hence S minus the sets $V(H) - \{w\}$ and $V(Q_j)$ for every j is a global offensive k-alliance of T' implying that

$$\gamma_o^k(T') \leq \gamma_o^k(T) - |V(H)| + 1 - \sum_{j=1}^{t} |V(Q_j)|,$$

and so

$$\gamma_o^k(T) = \gamma_o^k(T') + |V(H)| - 1 + \sum_{j=1}^{t} |V(Q_j)|.$$

Using the induction on T', we obtain $\gamma_o^k(T) = (n(T) + |L_k(T)|)/2$. □

We now give a constructive characterization of the trees T with the property that $\gamma_k(T) = (n(T) + |L_k(T)|)/2$ for every integer $k \geq 2$.

Theorem 2.5. *Let $k \geq 2$ be an integer. A tree T satisfies $\gamma_k(T) = (n(T) + |L_k(T)|)/2$ if and only if either $\Delta(T) \leq k-2$ or $T \in \mathcal{F}_k$.*

Proof. Clearly, if T is a tree with $\Delta(T) \leq k-2$, then $|L_k(T)| = n(T)$ and so $\gamma_k(T) = n(T) = (n(T) + |L_k(T)|)/2$. By Lemma 2.4, if $T \in \mathcal{F}_k$, then $\gamma_k(T) = (n(T) + |L_k(T)|)/2$.

Let us prove the necessity. Let T be a tree with $\gamma_o^k(T) = (n(T) + |L_k(T)|)/2$ for a positive integer $k \geq 2$. Suppose that $\Delta(T) \geq k-1$ and let $Z(T) = \{x \in V(T) :$

$\deg_T(x) \geq k-1\}$. We use an induction on the size of $Z(T)$, where $|Z(T)| \geq 1$. If $|Z(T)| = 1$ then T is an exact $\mathcal{N}_k$-tree and hence $T \in \mathcal{F}_k$, because otherwise $\gamma_o^k(T) = n(T) - 1 < n(T) - \frac{1}{2} = \frac{n(T)+|L_k(T)|}{2}$.

Let $|Z(T)| \geq 2$ and assume that every tree T' with $|Z(T')| < |Z(T)|$ such that $\gamma_o^k(T') = (n(T') + |L_k(T')|)/2$ is in $\mathcal{F}_k$.

Note that we have seen in the proof of Theorem 2.1 that $A \cup L_k(T)$ and $B \cup L_k(T)$ are two global offensive k-alliances of T, where $\min\{|A \cup L_k(T)|, |B \cup L_k(T)|\} \leq \frac{n(T)-|L_k(T)|}{2}$. It follows that if $\gamma_o^k(T) = \frac{n(T)+|L_k(T)|}{2}$, then $A \cup L_k(T)$ and $B \cup L_k(T)$ are two $\gamma_o^k(T)$-sets.

Let T be a tree with $\gamma_o^k(T) = (n(T)+|L_k(T)|)/2$ and S a $\gamma_o^k(T)$-set. If every vertex of T has degree at most $k-1$ then T is an exact $\mathcal{N}_k$-tree. So assume that $\Delta(T) \geq k$. Then T has at least two vertices of degree at least k for otherwise $\gamma_o^k(T) = n - 1 \neq (n(T) + |L_k(T)|)/2$ since $|L_k(T)| = n - 1$, a contradiction.

We now root T at a vertex r of maximum eccentricity. Let w be a vertex of degree at least k at maximum distance from r. Such a vertex exists since $\Delta(T) \geq k$. Clearly $w \neq r$ and T_w is an $\mathcal{N}_k$-tree. Let u be the parent of w in the rooted tree. Assume that $\deg_T(u) < k$. Without loss of generality we may assume that $w \in A$. Then $u \in L_k(T)$ and every descendant of w is in $L_k(T)$. As seen above $A \cup L_k(T)$ is a $\gamma_o^k(T)$-set but then $(A - \{w\}) \cup L_k(T)$ is a global offensive k-alliance of T, a contradiction. Thus $\deg_T(u) \geq k$. Likewise if u has a child $b \neq w$ of degree at least k, then $w, b \in A$, and so $(A - \{w, b\}) \cup \{u\} \cup L_k(T)$ is a global offensive k-alliance of T of size $\frac{n(T)-|L_k(T)|}{2} - 1$ which leads to a contradiction too. Thus every child of u besides w has degree at most $k-1$ and so every vertex of $D(u) - \{w\}$ has degree at most $k-1$. We distinguish between two cases:

Case 1. Assume that $\deg_T(w) \geq k+2$. Assume that $\deg_T(u) \geq k+2$. Then every neighbor of u is in $L_k(T)$ or in A (w and possibly the parent of u). It follows that $(A - \{w\}) \cup L_k(T)$ is a global offensive k-alliance of T, a contradiction.

It remains the case that $k \leq \deg_T(u) \leq k+1$. Now consider the subtree $T' = T - (T_w \cup T_b)$, where T_b is any subtree rooted at a child $b \neq w$ of u if $\deg_T(u) = k+1$ and $V(T_b) = \emptyset$ if $\deg_T(u) = k$. Thus in both cases u has degree $k-1$ in T' and b has degree at most $k-2$ in T_b. Then every $\gamma_o^k(T')$-set contains u and such a set can be extended to a global offensive k-alliance of T by adding $(V(T_w) - \{w\}) \cup V(T_b)$, and so $\gamma_o^k(T) \leq \gamma_o^k(T') + |D(w)| + |D[b]|$. The equality is obtained by the fact that $(B \cup L_k(T)) - (D(w) \cup D[b])$ is a global offensive k-alliance of T'. Since w is a vertex of degree at least k at maximum distance from r, we deduce that $|L_k(T)| = |L_k(T')| + |D(w)| + |D[b]| - 1$. It follows that

$$\frac{n(T) + |L_k(T)|}{2} = \gamma_o^k(T) = \gamma_o^k(T') + |D(w)| + |D[b]|$$

and therefore $\frac{n(T')+|L_k(T')|}{2} = \gamma_o^k(T')$. Since $|Z(T')| < |Z(T)|$, by induction on T', we have $T' \in \mathcal{F}_k$. Because T is obtained from T' by using Operation $\mathcal{O}_1$, $T \in \mathcal{F}_k$.

Case 2. Assume that $k \leq \deg_T(w) \leq k+1$. Let $C(u) = \{w, y_1, \ldots, y_p\}$ where $p = \deg_T(u) - 2$. Recall that every vertex of $C(u) - \{w\}$ has degree at most $k-1$. Let

$T' = T - T_w - \bigcup_{j=1}^{p+2-k} T_{y_j}$. Then T' is nontrivial and $\deg_{T'}(u) = k-1$. It can be seen that

$$\gamma_o^k(T) = \gamma_o^k(T') + \left| D(w) \cup \left(\bigcup_{j=1}^{p+2-k} D[y_j] \right) \right|,$$

$$n(T) = n(T') + \left| D(w) \cup \left(\bigcup_{j=1}^{p+2-k} D[y_j] \right) \right| + 1$$

and

$$L_k(T) = L_k(T') + \left| D(w) \cup \left(\bigcup_{j=1}^{p+2-k} D[y_j] \right) \right| - 1,$$

implying that $\gamma_o^k(T') = (n(T') + |L_k(T')|)/2$ with $|Z(T')| < |Z(T)|$. By the inductive hypothesis on T', we have $T' \in \mathcal{F}_k$. Thus $T \in \mathcal{F}_k$ because it is obtained from T' by using Operation $\mathcal{O}_2$. $\square$

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