ON THE EXISTENCE OF POSITIVE CONTINUOUS SOLUTIONS FOR SOME POLYHARMONIC ELLIPTIC SYSTEMS ON THE HALF SPACE

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Abstract. We study the existence of positive continuous solutions of the nonlinear polyharmonic system $(-\Delta)^m u + \lambda q g(v) = 0, (-\Delta)^m v + \mu p f(u) = 0$ in the half space $\mathbb{R}^n_+ := \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : x_n > 0\}$, where $m \geq 1$ and $n > 2m$. The nonlinear term is required to satisfy some conditions related to the Kato class $K^\infty_{m,n}(\mathbb{R}^n_+)$. Our arguments are based on potential theory tools associated to $(-\Delta)^m$ and properties of functions belonging to $K^\infty_{m,n}(\mathbb{R}^n_+)$.

Keywords: polyharmonic elliptic system, Green function, Kato class, positive continuous solution, Schauder fixed point theorem.

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1. INTRODUCTION

Let m be a positive integer and $\mathbb{R}^n_+ = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_n > 0\}$, where $n > 2m$. An explicit expression for the Green function $G_{m,n}$ of $(-\Delta)^m$ on $\mathbb{R}^n_+$, with Dirichlet boundary conditions $(\frac{\partial}{\partial x_n})^j u = 0$, $0 \leq j \leq m-1$ was given in [4] by

$$G_{m,n}(x,y) = k_{m,n} |\, x - y \,|^{2m-n} \int_1^{\frac{|x-\overline{y}|}{|x-y|}} \frac{(v^2-1)^{m-1}}{v^{n-1}} dv, \quad x, y \in \mathbb{R}^n_+,$$

$k_{m,n}$ is a positive constant and $\overline{y} = (y_1, y_2, \ldots, y_{n-1}, -y_n)$.

Since the Green function $G_{m,n}$ is positive and based on the potential theory approach, we investigate in this paper the existence of positive continuous solutions (in the sense of distributions) for the following polyharmonic elliptic system

$$\begin{cases} (-\Delta)^m u + \lambda p f(v) = 0 \text{ in } \mathbb{R}^n_+, \\ (-\Delta)^m v + \mu q g(u) = 0 \text{ in } \mathbb{R}^n_+, \\ \lim_{x\to(\xi,0)} \frac{u(x)}{x_n^{m-1}} = a\varphi(\xi), \forall\, \xi \in \mathbb{R}^{n-1}, \\ \lim_{x_n\to+\infty} \frac{u(x)}{x_n^m} = \alpha, \\ \lim_{x\to(\xi,0)} \frac{v(x)}{x_n^{m-1}} = b\psi(\xi), \forall\, \xi \in \mathbb{R}^{n-1}, \\ \lim_{x_n\to+\infty} \frac{v(x)}{x_n^m} = \beta, \end{cases} \tag{1.1}$$

where λ, μ are nonnegative constants, a, b, α and β are nonnegative constants such that $a + \alpha > 0$, $b + \beta > 0$ and the functions φ and ψ are non-trivial nonnegative bounded continuous functions on $\partial\mathbb{R}^n_+ := \{(x_1, x_2, \ldots, x_{n-1}, 0) \in \mathbb{R}^n\}$ which we identify to $\mathbb{R}^{n-1}$.

In a recent paper [14], we have treated a similar polyharmonic problem in the unit ball $B = \{x \in \mathbb{R}^n : |x| < 1\}$ of $\mathbb{R}^n$ $(n \geq 2)$.

For the case $m = 1$, the existence of solutions for nonlinear elliptic systems has been extensively studied for both bounded and unbounded $C^{1,1}$domains in $\mathbb{R}^n$ $(n \geq 3)$ see for example [7–13, 16, 18].

For our study we use closely the following interesting estimates for $G_{m,n}$, which were established in [4]. For each x, $y \in \mathbb{R}^n_+$

$$G_{m,n}(x, y) \approx \frac{1}{|x-y|^{n-2m}} \min\Big(1, \frac{(x_n y_n)^m}{|x-y|^{2m}}\Big). \tag{1.2}$$

Here and throughout the paper for nonnegative functions f and g on a set S, the notation $f \approx g$ means that there exists a constant $c > 0$ such that $\frac{1}{c} g \leq f \leq cg$ on S.

From (1.2), Bachar *et al.* [4] derived the following 3G-inequality.

Theorem 1.1. *There exists $C_{m,n} > 0$ such that for each $x, y, z \in \mathbb{R}^n_+$*

$$\frac{G_{m,n}(x,z)G_{m,n}(z,y)}{G_{m,n}(x,y)} \leq C_{m,n}\Big[\Big(\frac{z_n}{x_n}\Big)^m G_{m,n}(x,z) + \Big(\frac{z_n}{y_n}\Big)^m G_{m,n}(y,z)\Big]. \tag{1.3}$$

Using these estimates, the authors in [4] introduce a large class of functions called the Kato class and denoted by $K^\infty_{m,n}(\mathbb{R}^n_+) := K^\infty_{m,n}$, defined as follows.

Definition 1.2 ([4])**.** A Borel measurable function q in $\mathbb{R}^n_+$ belongs to the Kato class $K^\infty_{m,n}$ if q satisfies

$$\lim_{\alpha\to 0}\Bigg(\sup_{x\in\mathbb{R}^n_+} \int_{\mathbb{R}^n_+\cap B(x,\alpha)} \Big(\frac{y_n}{x_n}\Big)^m G_{m,n}(x,y)|q(y)|dy\Bigg) = 0 \tag{1.4}$$

and

$$\lim_{M\to\infty}\Bigg(\sup_{x\in\mathbb{R}^n_+} \int_{\mathbb{R}^n_+\cap(|y|\geq M)} \Big(\frac{y_n}{x_n}\Big)^m G_{m,n}(x,y)|q(y)|dy\Bigg) = 0. \tag{1.5}$$

To illustrate, we cite as a typical example of functions belonging to the class $K_{m,n}^{\infty}$ the following example.

Example 1.3 ([4]). Let λ, $\mu \in \mathbb{R}$ and $q(x) = \frac{1}{(1+|x|)^{\mu-\lambda} x_n^{\lambda}}$ for $x \in \mathbb{R}_+^n$. Then the function $q \in K_{m,n}^{\infty}$ if and only if $\lambda < 2m < \mu$.

We note that for $m = 1$, the corresponding elliptic class $K^{\infty}\left(\mathbb{R}_+^n\right) := K_{1,n}^{\infty}\left(\mathbb{R}_+^n\right)$ has been studied by Bachar and Mâagli in [1] for $n \geq 3$ and by Bachar *et al.* in [2] for $n = 2$.

The class $K_{m,n}^{\infty}$ was fully developed and exploited to study the existence of positive continuous solutions for some polyharmonic nonlinear elliptic problems (see [4, 5]).

Before presenting our main results, we give some notations and terminology to be used throughout the paper. We set θ the harmonic function defined on $\mathbb{R}_+^n$ by $\theta(x) = x_n$. For any nonnegative continuous bounded function φ on $\mathbb{R}^{n-1}$, we denote by $H\varphi$ the unique harmonic bounded function in $\mathbb{R}_+^n$ satisfying

$$\lim_{x \to (\xi,0)} H\varphi(x) = \varphi(\xi), \quad \forall\, \xi \in \mathbb{R}^{n-1}. \tag{1.6}$$

We remark that the function $x \mapsto (\theta(x))^{m-1} H\varphi(x)$ is a classical solution of the problem

$$\begin{cases} (-\Delta)^m u = 0 & \text{in } \mathbb{R}_+^n, \\ \lim_{x \to (\xi,0)} \frac{u(x)}{x_n^{m-1}} = \varphi(\xi), & \forall \xi \in \mathbb{R}^{n-1}. \end{cases}$$

We also refer to Vf the m-potential of a measurable nonnegative function f on $\mathbb{R}_+^n$, defined by

$$Vf(x) = \int_{\mathbb{R}_+^n} G_{m,n}(x,y) f(y) dy \quad \text{for } x \in \mathbb{R}_+^n.$$

As in the classical case the following assertions are equivalent for each nonnegative measurable function f on $\mathbb{R}_+^n$:

(i) $Vf \neq \infty$, and consequently $Vf \in L^1_{loc}(\mathbb{R}_+^n)$,

(ii) $\int_{\mathbb{R}_+^n} \frac{y_n^m}{(1+|y|)^n} f(y) dy < \infty$.

Hence for each nonnegative measurable function f on $\mathbb{R}_+^n$ such that $Vf \in L^1_{loc}(\mathbb{R}_+^n)$, we have

$$(-\Delta)^m (Vf) = f \text{ (in the distributional sense)}.$$

As usual, we denote

$$C\left(\mathbb{R}_+^n\right) = \{w : \mathbb{R}_+^n \to \mathbb{R},\ w \text{ is continuous}\},$$

$$C_0(\mathbb{R}_+^n) = \left\{ w \in C\left(\mathbb{R}_+^n\right), \lim_{x_n \to 0} w(x) = 0 \text{ and } \lim_{|x| \to \infty} w(x) = 0 \right\}$$

and

$$C_b(\mathbb{R}_+^n) = \{w \in C\left(\mathbb{R}_+^n\right), w \text{ is bounded in } \mathbb{R}_+^n\}.$$

Our paper is organized as follows. In Section 2 we recall some properties of functions belonging to the Kato class $K^{\infty}_{m,n}$ developed in [4]. Next, we present a subclass of m-potential functions which allows us to establish the following result which is a key tool in our study.

Theorem 1.4. *Let $\beta \in [m-1, m)$, $q \in K^{\infty}_{m,n}$. The function v defined on $\mathbb{R}^n_+$ by*

$$v(x) = \int_{\mathbb{R}^n_+} \left(\frac{y_n}{x_n}\right)^{\beta} G_{m,n}(x,y) q(y) dy$$

is in $C_0(\mathbb{R}^n_+)$.

Remark 1.5. For $\beta = m$, the authors in [4] showed that the function v given in Theorem 1.4, is continuous in $\overline{\mathbb{R}^n_+}$ and satisfies $\lim\limits_{|x|\to\infty} v(x) = 0$.

As mentioned above, the main goal of this paper is to prove two existence results for the system (1.1), stated in Theorem 1.6 and Theorem 1.7 below and proved in Sections 3 and 4. Section 5 is reserved to examples.

For our first existence result, we assume the following hypotheses:

(H_1) The functions f, $g : (0,\infty) \to [0,\infty)$ are continuous and nondecreasing.

(H_2) The functions p, q are nonnegative measurable on $\mathbb{R}^n_+$ and for each $c > 0$, the functions

$$p_c := \frac{p}{\theta^{m-1}} f(c\theta^{m-1}(\theta+1)), \quad q_c := \frac{q}{\theta^{m-1}} g(c\theta^{m-1}(\theta+1))$$

belong to the Kato class $K^{\infty}_{m,n}$.

(H_3) We suppose that

$$\lambda_0 = \inf_{x\in\mathbb{R}^n_+} \frac{\alpha x_n^m + a x_n^{m-1} H\varphi(x)}{V(pf(\beta\theta^m + b\theta^{m-1}H\psi))(x)} > 0,$$

$$\mu_0 = \inf_{x\in\mathbb{R}^n_+} \frac{\beta x_n^m + b x_n^{m-1} H\psi(x)}{V(qg(\alpha\theta^m + a\theta^{m-1}H\varphi))(x)} > 0.$$

Using an iterative scheme, we obtain the following theorem.

Theorem 1.6. *Assume (H_1)–(H_3). Then for each $\lambda \in [0, \lambda_0)$ and $\mu \in [0, \mu_0)$, problem (1.1) has a positive continuous solution (u, v) such that*

$$\begin{cases} (1 - \frac{\lambda}{\lambda_0})(\alpha\theta^m + a\theta^{m-1}H\varphi) \le u \le \alpha\theta^m + a\theta^{m-1}H\varphi, \\ (1 - \frac{\mu}{\mu_0})(\beta\theta^m + b\theta^{m-1}H\psi) \le v \le \beta\theta^m + b\theta^{m-1}H\psi. \end{cases}$$

Our second existence result deals with problem (1.1) when the functions f, g are continuous and nonincreasing, $\lambda = \mu = a = b = 1$ and α, β are nonnegative constants.

More precisely, we fix a non-trivial nonnegative bounded continuous function Φ on $\mathbb{R}^{n-1}$ and we need the following assumptions:

(H_4) The functions $f, g : (0, \infty) \to [0, \infty)$ are continuous and nonincreasing.
(H_5) The functions p, q are nonnegative measurable on $\mathbb{R}^n_+$ such that the functions

$$\widetilde{p} := p\,\frac{f(\theta^{m-1} H\Phi)}{\theta^{m-1} H\Phi}, \quad \widetilde{q} := q\,\frac{g(\theta^{m-1} H\Phi)}{\theta^{m-1} H\Phi}$$

belong to the Kato class $K^{\infty}_{m,n}$.

Using a fixed point argument, we obtain the following theorem.

Theorem 1.7. *Assume that $\lambda = \mu = a = b = 1$ and that* (H_4)–(H_5) *are satisfied. Suppose that there exists $\gamma > 1$ such that $\varphi \geq \gamma\Phi$ and $\psi \geq \gamma\Phi$ on $\mathbb{R}^{n-1}$. Then problem* (1.1) *has a positive continuous solution (u, v) satisfying*

$$\begin{cases} \alpha\theta^m + \theta^{m-1} H\Phi \leq u \leq \alpha\theta^m + \theta^{m-1} H\varphi, \\ \beta\theta^m + \theta^{m-1} H\Phi \leq u \leq \beta\theta^m + \theta^{m-1} H\psi. \end{cases} \tag{1.7}$$

Throughout the paper the letter c denotes a generic positive constant which may vary from line to line.

2. MODULUS OF CONTINUITY

We collect in the following some preliminary results useful for our study. For the proofs we refer to [4, 5].

Proposition 2.1. *Let $q \in K^{\infty}_{m,n}$. Then:*

(i) *The function $x \mapsto \frac{x_n^{2m-1}}{(1+|x|)^n} q(x)$ is in $L^1(\mathbb{R}^n_+)$.*
In particular the function $x \mapsto x_n^{2m-1} q(x)$ is in $L^1_{loc}(\overline{\mathbb{R}^n_+})$.
(ii) $\alpha_q := \sup\limits_{x,y \in \mathbb{R}^n_+} \int_{\mathbb{R}^n_+} \frac{G_{m,n}(x,z) G_{m,n}(z,y)}{G_{m,n}(x,y)} |q(z)|\, dz < \infty$.

Moreover, for each nonnegative harmonic function h in $\mathbb{R}^n_+$ we have for $x \in \mathbb{R}^n_+$,

$$\int\limits_{\mathbb{R}^n_+} G_{m,n}(x,y) y_n^{m-1} h(y) |q(y)| dy \leq \alpha_q x_n^{m-1} h(x). \tag{2.1}$$

Proposition 2.2. *Let $x_0 \in \overline{\mathbb{R}^n_+}$, then for each $q \in K^{\infty}_{m,n}$, we have*

$$\lim_{\alpha \to 0} \Big(\sup_{x \in \mathbb{R}^n_+} \int\limits_{\mathbb{R}^n_+ \cap B(x_0,\alpha)} \Big(\frac{y_n}{x_n}\Big)^m G_{m,n}(x,y) |q(y)| dy \Big) = 0. \tag{2.2}$$

Now, we provide a subclass of m-potential functions.

Proposition 2.3. *Let q be the function defined on $\mathbb{R}^n_+$ by*

$$q(x) = \frac{1}{x_n^\lambda}, \quad m < \lambda < m+1.$$

Then there exists a constant $c_{m,n,\lambda} > 0$ such that for each $x \in \mathbb{R}^n_+$

$$Vq(x) = c_{m,n,\lambda}\, x_n^{2m-\lambda}.$$

Proof. Let $\lambda \in (m, m+1)$ and $x \in \mathbb{R}^n_+$.

$$Vq(x) = \int_{\mathbb{R}^n_+} G_{m,n}(x,y)q(y)dy = k_{m,n} \int_{\mathbb{R}^n_+} \frac{|x-y|^{2m-n}}{y_n^\lambda} \int_1^{\frac{|x-\overline{y}|}{|x-y|}} \frac{(v^2-1)^{m-1}}{v^{n-1}} dvdy.$$

Putting

$$|x-\overline{y}|^2 = |x'-y'|^2 + (x_n+y_n)^2$$

and

$$|x-y|^2 = |x'-y'|^2 + (x_n-y_n)^2.$$

Then, by the change of variable $r = |x'-y'|$, we obtain

$$Vq(x) = k_{m,n} \int_0^{+\infty}\int_0^{+\infty} \frac{\left(r^2+(x_n-y_n)^2\right)^{\frac{2m-n}{2}}}{y_n^\lambda} r^{n-2} \int_1^{\left(\frac{r^2+(x_n+y_n)^2}{r^2+(x_n-y_n)^2}\right)^{\frac{1}{2}}} \frac{(v^2-1)^{m-1}}{v^{n-1}} dvdy_ndr,$$

which implies, by using the transformations $t = \frac{y_n}{x_n}$ and $s = \frac{r}{x_n}$, that

$$Vq(x) = k_{m,n}x_n^{2m-\lambda} \int_0^{+\infty}\int_0^{+\infty} \frac{s^{n-2}}{t^\lambda}(s^2+(1-t)^2)^{\frac{2m-n}{2}} \int_1^{\left(\frac{s^2+(1+t)^2}{s^2+(1-t)^2}\right)^{\frac{1}{2}}} \frac{(v^2-1)^{m-1}}{v^{n-1}} dvdtds.$$

Finally, making the change of variable $u = v^2-1$, we obtain

$$Vq(x) = \frac{k_{m,n}}{2}x_n^{2m-\lambda} \int_0^{+\infty}\int_0^{+\infty} \frac{s^{n-2}}{t^\lambda}\left(s^2+(1-t)^2\right)^{\frac{2m-n}{2}} \int_0^{\frac{4t}{s^2+(1-t)^2}} \frac{u^{m-1}}{(1+u)^{\frac{n}{2}}} dudtds.$$

To achieve the desired result, we claim that

$$\int_0^{+\infty}\int_0^{+\infty} \frac{s^{n-2}}{t^\lambda}\left(s^2+(1-t)^2\right)^{\frac{2m-n}{2}} \int_0^{\frac{4t}{s^2+(1-t)^2}} \frac{u^{m-1}}{(1+u)^{\frac{n}{2}}} dudtds$$

converges. Indeed, we note that for $0 < m < \frac{n}{2}$

$$\int_0^{\frac{4t}{s^2+(1-t)^2}} \frac{u^{m-1}}{(1+u)^{\frac{n}{2}}} du \approx \min\left\{1, \left(\frac{4t}{s^2+(1-t)^2}\right)^m\right\}$$

and for $m < \lambda < m+1$, we have

$$\int_0^{+\infty}\int_0^{+\infty} \frac{s^{n-2}}{t^\lambda}\left(s^2+(1-t)^2\right)^{\frac{2m-n}{2}} \min\left\{1, \left(\frac{4t}{s^2+(1-t)^2}\right)^m\right\} dtds$$

converges. Then the claim is proved. This ends the proof. □

Proposition 2.4. *Let $m-1 \le \beta < m, x_0 \in \overline{\mathbb{R}^n_+}$. Then for each $q \in K^\infty_{m,n}$*

$$\lim_{\alpha\to 0}\left(\sup_{x\in\mathbb{R}^n_+} \int_{\mathbb{R}^n_+\cap B(x_0,\alpha)} \left(\frac{y_n}{x_n}\right)^\beta G_{m,n}(x,y)|q(y)|dy\right) = 0, \tag{2.3}$$

$$\lim_{M\to+\infty}\left(\sup_{x\in\mathbb{R}^n_+} \int_{\mathbb{R}^n_+\cap\{|y|\ge M\}} \left(\frac{y_n}{x_n}\right)^\beta G_{m,n}(x,y)|q(y)|dy\right) = 0. \tag{2.4}$$

Proof. For $\beta = m-1$, the results were proved in [4]. For $\beta \in (m-1, m)$, we deduce from Proposition 2.3, that there exists a constant $c_{m,n,\beta} > 0$ such that

$$x_n^\beta = c_{m,n,\beta} \int_{\mathbb{R}^n_+} \frac{G_{m,n}(x,z)}{z_n^{2m-\beta}} dz, \quad x \in \mathbb{R}^n_+.$$

Now, let $\alpha > 0$, then by Fubini's theorem and (1.3), we have

$$\begin{aligned}
&\int_{\mathbb{R}^n_+\cap B(x_0,\alpha)} y_n^\beta G_{m,n}(x,y)|q(y)|dy = \\
&= c_{m,n,\beta} \int_{\mathbb{R}^n_+\cap B(x_0,\alpha)} \int_{\mathbb{R}^n_+} \frac{G_{m,n}(y,z)}{z_n^{2m-\beta}} G_{m,n}(x,y)|q(y)|dzdy = \\
&= c_{m,n,\beta} \int_{\mathbb{R}^n_+}\left(\int_{\mathbb{R}^n_+\cap B(x_0,\alpha)} \frac{G_{m,n}(x,y)G_{m,n}(y,z)}{G_{m,n}(x,z)}|q(y)|dy\right)\frac{G_{m,n}(x,z)}{z_n^{2m-\beta}} dz \le \\
&\le c\Big(\sup_{\xi\in\mathbb{R}^n_+} \int_{\mathbb{R}^n_+\cap B(x_0,\alpha)} \left(\frac{y_n}{\xi_n}\right)^m G_{m,n}(\xi,y)|q(y)|dy\Big) x_n^\beta,
\end{aligned}$$

which implies (2.3) by dividing by x_n^β and using (2.2).

Using (1.3) and (1.5), we obtain (2.4) by similar arguments. □

Proof of Theorem 1.4. Let $\beta \in [m-1, m)$, $x_0 \in \overline{\mathbb{R}^n_+}$ and $\varepsilon > 0$. By Proposition 2.4, there exist $\alpha > 0$ and $M > 0$ such that

$$\sup_{\xi \in \mathbb{R}^n_+} \int\limits_{\mathbb{R}^n_+ \cap B(x_0, 2\alpha)} \left(\frac{y_n}{\xi_n}\right)^\beta G_{m,n}(\xi, y)|q(y)|dy \leq \varepsilon \tag{2.5}$$

and

$$\sup_{\xi \in \mathbb{R}^n_+} \int\limits_{\mathbb{R}^n_+ \cap \{|y| \geq M\}} \left(\frac{y_n}{\xi_n}\right)^\beta G_{m,n}(\xi, y)|q(y)|dy \leq \varepsilon. \tag{2.6}$$

First, we aim to prove that $v \in C\left(\mathbb{R}^n_+\right)$. We fix $x_0 \in \mathbb{R}^n_+$. Let $x, z \in \mathbb{R}^n_+ \cap B(x_0, \alpha)$. It follows from (2.5) and (2.6) that

$$\begin{aligned}
|v(x) - v(z)| \leq \int\limits_{\mathbb{R}^n_+} \left| \frac{G_{m,n}(x,y)}{x_n^\beta} - \frac{G_{m,n}(z,y)}{z_n^\beta} \right| y_n^\beta |q(y)|dy \leq \\
\leq 2 \sup_{\xi \in \mathbb{R}^n_+} \int\limits_{\mathbb{R}^n_+ \cap B(x_0, 2\alpha)} \left(\frac{y_n}{\xi_n}\right)^\beta G_{m,n}(\xi, y)|q(y)|dy + \\
+ 2 \sup_{\xi \in \mathbb{R}^n_+} \int\limits_{\mathbb{R}^n_+ \cap (|y| \geq M)} \left(\frac{y_n}{\xi_n}\right)^\beta G_{m,n}(\xi, y)|q(y)|dy + \\
+ \int\limits_{\mathbb{R}^n_+ \cap B^c(x_0, 2\alpha) \cap B(0,M)} \left| \frac{G_{m,n}(x,y)}{x_n^\beta} - \frac{G_{m,n}(z,y)}{z_n^\beta} \right| y_n^\beta |q(y)|dy \leq \\
\leq 4\varepsilon + \int\limits_{\mathbb{R}^n_+ \cap B^c(x_0, 2\alpha) \cap B(0,M)} \left| \frac{G_{m,n}(x,y)}{x_n^\beta} - \frac{G_{m,n}(z,y)}{z_n^\beta} \right| y_n^\beta |q(y)|dy.
\end{aligned}$$

If $|y - x_0| \geq 2\alpha$, then $|y - x| \geq \alpha$ and $|y - z| \geq \alpha$.

So applying (1.2) for $y \in \mathbb{R}^n_+ \cap B^c(x_0, 2\alpha) \cap B(0, M)$, we have

$$\begin{aligned}
\left| \frac{G_{m,n}(x,y)}{x_n^\beta} - \frac{G_{m,n}(z,y)}{z_n^\beta} \right| y_n^\beta |q(y)| \leq c \left(\frac{x_n^{m-\beta}}{|x-y|^n} + \frac{z_n^{m-\beta}}{|x-z|^n} \right) y_n^{m+\beta} |q(y)| \leq \\
\leq c y_n^{2m-1} |q(y)|.
\end{aligned}$$

On the other hand for $y \in \mathbb{R}^n_+ \cap B^c(x_0, 2\alpha) \cap B(0, M)$, the function $x \mapsto \frac{G_{m,n}(x,y)}{x_n^\beta}$ is continuous in $\mathbb{R}^n_+ \cap B(x_0, \alpha)$. Since $q \in K^\infty_{m,n}$ we deduce by Proposition 2.1 (i) that the function $x \mapsto x_n^{2m-1} q(x)$ is in $L^1_{loc}\left(\overline{\mathbb{R}^n_+}\right)$ and so by the dominated convergence theorem, we obtain that

$$\int\limits_{\mathbb{R}^n_+ \cap B^c(x_0, 2\alpha) \cap B(0,M)} \left| \frac{G_{m,n}(x,y)}{x_n^\beta} - \frac{G_{m,n}(z,y)}{z_n^\beta} \right| y_n^\beta |q(y)|dy \to 0 \quad \text{as} \quad |x - z| \to 0.$$

Thus we deduce that v is continuous on $\mathbb{R}^n_+$.

Now, let $x_0 = (\xi, 0), \xi \in \mathbb{R}^{n-1}$. We shall show that

$$\lim_{x \to (\xi,0)} v(x) = 0.$$

Let $x \in B(x_0, \alpha) \cap \mathbb{R}^n_+$, then we have by (2.5) and (2.6)

$$\begin{aligned} 0 \le v(x) \le \sup_{\xi \in \mathbb{R}^n_+} & \int\limits_{\mathbb{R}^n_+ \cap B(x_0, 2\alpha)} \left(\frac{y_n}{\xi_n}\right)^\beta G_{m,n}(\xi, y)|q(y)|dy+ \\ & + \sup_{\xi \in \mathbb{R}^n_+} \int\limits_{\mathbb{R}^n_+ \cap (|y| \ge M)} \left(\frac{y_n}{\xi_n}\right)^\beta G_{m,n}(\xi, y)|q(y)|dy+ \\ & + \int\limits_{\mathbb{R}^n_+ \cap B^c(x_0, 2\alpha) \cap B(0,M)} \left(\frac{y_n}{x_n}\right)^\beta G_{m,n}(x, y)|q(y)|dy \le \\ & \le 2\varepsilon + \int\limits_{\mathbb{R}^n_+ \cap B^c(x_0, 2\alpha) \cap B(0,M)} \left(\frac{y_n}{x_n}\right)^\beta G_{m,n}(x, y)|q(y)|dy. \end{aligned}$$

For $y \in \mathbb{R}^n_+ \cap B^c(x_0, 2\alpha)$ we have $|y - x| \ge \alpha$. So from (1.2) we get

$$\begin{aligned} & \int\limits_{\mathbb{R}^n_+ \cap B^c(x_0, 2\alpha) \cap B(0,M)} \left(\frac{y_n}{x_n}\right)^\beta G_{m,n}(x, y)|q(y)|dy \le \\ & \le c x_n^{m-\beta} \int\limits_{\mathbb{R}^n_+ \cap B^c(x_0, 2\alpha) \cap B(0,M)} \frac{y_n^{m+\beta}}{|x-y|^n} |q(y)|dy \le \\ & \le c x_n^{m-\beta} \int\limits_{\mathbb{R}^n_+ \cap B^c(x_0, 2\alpha) \cap B(0,M)} y_n^{2m-1} |q(y)|dy, \end{aligned}$$

which implies by Proposition 2.1 (i) that

$$\int\limits_{\mathbb{R}^n_+ \cap B^c(x_0, 2\alpha) \cap B(0,M)} \left(\frac{y_n}{x_n}\right)^\beta G_{m,n}(x, y)|q(y)|dy \le c x_n^{m-\beta}.$$

Hence, we get

$$\int\limits_{\mathbb{R}^n_+ \cap B^c(x_0, 2\alpha) \cap B(0,M)} \left(\frac{y_n}{x_n}\right)^\beta G_{m,n}(x, y)|q(y)|dy \to 0 \text{ as } x \to (\xi, 0).$$

So, we deduce that $v(x) \to 0$ as $x \to (\xi, 0)$.

Finally, we intend to show that

$$\lim_{|x|\to\infty} v(x) = 0.$$

Let $x \in \mathbb{R}^n_+$ such that $|x| \geq M+1$. By (2.6), we have

$$v(x) \leq \sup_{\xi\in\mathbb{R}^n_+} \int\limits_{\mathbb{R}^n_+\cap(|y|\geq M)} \left(\frac{y_n}{\xi_n}\right)^\beta G_{m,n}(\xi,y)|q(y)|dy+$$

$$+ \int\limits_{\mathbb{R}^n_+\cap B(0,M)} \left(\frac{y_n}{x_n}\right)^\beta G_{m,n}(x,y)|q(y)|dy \leq$$

$$\leq \varepsilon + \int\limits_{\mathbb{R}^n_+\cap B(0,M)} \left(\frac{y_n}{x_n}\right)^\beta G_{m,n}(x,y)|q(y)|dy.$$

Now, for $y \in \mathbb{R}^n_+ \cap B(0,M)$, we obtain by (1.2)

$$\left(\frac{y_n}{x_n}\right)^\beta G_{m,n}(x,y)|q(y)| \leq c\frac{x_n^{m-\beta} y_n^{m+\beta}}{|x-y|^n}|q(y)| \leq$$

$$\leq c\frac{|x|^{m-\beta}}{(|x|-M)^n} y_n^{2m-1}|q(y)| \leq c\frac{|x|}{(|x|-M)^n} y_n^{2m-1}|q(y)|.$$

Hence, using Proposition 2.1 (i), we get that $v(x) \to 0$ as $|x| \to \infty$.

This ends the proof. □

By similar arguments as in the proof of Theorem 1.4, we prove the following proposition.

Proposition 2.5. *Let $m-1 \leq \beta < m$. For any nonnegative function $q \in K^\infty_{m,n}$, the family of functions*

$$\left\{ \int\limits_{\mathbb{R}^n_+} \left(\frac{y_n}{x_n}\right)^\beta G_{m,n}(x,y)\xi(y)dy,\ \ |\xi| \leq q \right\}$$

is relatively compact in $C_0(\mathbb{R}^n_+)$.

3. PROOF OF THEOREM 1.6

An important property about potential functions is given in the following lemma.

Lemma 3.1. *If f and g are nonnegative measurable functions defined on $\mathbb{R}^n_+$ such that $g \leq f$ and Vf is continuous in $\mathbb{R}^n_+$. Then Vg is also continuous in $\mathbb{R}^n_+$.*

Proof. Let θ be a nonnegative measurable function on $\mathbb{R}^n_+$ such that $f = g + \theta$. It is obvious that $V\theta$ and Vg are lower semi-continuous in $\mathbb{R}^n_+$ and $V\theta$ is finite. Thus, since $Vg = Vf - V\theta$, we conclude that Vg is continuous in $\mathbb{R}^n_+$. □

Proof of Theorem 1.6. Assume that the hypotheses (H_1)–(H_3) are satisfied. Then for each $x \in \mathbb{R}^n_+$, we have

$$\lambda_0 V\left(pf(\beta\theta^m + b\theta^{m-1}H\psi)\right)(x) \le \alpha\left(\theta(x)\right)^m + a\left(\theta(x)\right)^{m-1}H\varphi(x) \tag{3.1}$$

and

$$\mu_0 V\left(qg(\alpha\theta^m + a\theta^{m-1}H\varphi)\right)(x) \le \beta\left(\theta(x)\right)^m + b\left(\theta(x)\right)^{m-1}H\psi(x). \tag{3.2}$$

Let $\lambda \in [0, \lambda_0)$ and $\mu \in [0, \mu_0)$. We define the sequences $(u_k)_{k\ge 0}$ and $(v_k)_{k\ge 0}$ by

$$\begin{cases} v_0 = \beta\theta^m + b\theta^{m-1}H\psi > 0, \\ u_k = \alpha\theta^m + a\theta^{m-1}H\varphi - \lambda V(pf(v_k)), \\ v_{k+1} = \beta\theta^m + b\theta^{m-1}H\psi - \mu V(qg(u_k)). \end{cases}$$

We intend to prove that for all $k \in \mathbb{N}$,

$$0 < \left(1 - \frac{\lambda}{\lambda_0}\right)(\alpha\theta^m + a\theta^{m-1}H\varphi) \le u_k \le u_{k+1} \le \alpha\theta^m + a\theta^{m-1}H\varphi, \tag{3.3}$$

and

$$0 < \left(1 - \frac{\mu}{\mu_0}\right)(\beta\theta^m + b\theta^{m-1}H\psi) \le v_{k+1} \le v_k \le \beta\theta^m + b\theta^{m-1}H\psi. \tag{3.4}$$

For $k = 0$,

$$u_0 = \alpha\theta^m + a\theta^{m-1}H\varphi - \lambda V(pf(v_0)).$$

From (3.1) we have

$$\begin{aligned} u_0 &\ge \alpha\theta^m + a\theta^{m-1}H\varphi - \frac{\lambda}{\lambda_0}(\alpha\theta^m + a\theta^{m-1}H\varphi) \ge \\ &\ge \left(1 - \frac{\lambda}{\lambda_0}\right)(\alpha\theta^m + a\theta^{m-1}H\varphi) > 0. \end{aligned}$$

So,

$$v_1 - v_0 = -\mu V(qg(u_0)) \le 0.$$

On the other hand, since f is nondecreasing we have

$$u_1 - u_0 = \lambda V\left[p\left(f(v_0) - f(v_1)\right)\right] \ge 0.$$

Now, since $v_0 > 0$, then $u_0 \le \alpha\theta^m + a\theta^{m-1}H\varphi$ and using that g is nondecreasing and inequality (3.2) we get,

$$v_1 = \beta\theta^m + b\theta^{m-1}H\psi - \mu V(qg(u_0)) \ge \left(1 - \frac{\mu}{\mu_0}\right)(\beta\theta^m + b\theta^{m-1}H\psi) > 0.$$

This together with the fact that f is nondecreasing imply that

$$u_1 \leq \alpha\theta^m + a\theta^{m-1}H\varphi.$$

Finally, we deduce

$$\begin{cases} 0 < \left(1 - \frac{\lambda}{\lambda_0}\right)(\alpha\theta^m + a\theta^{m-1}H\varphi) \leq u_0 \leq u_1 \leq \alpha\theta^m + a\theta^{m-1}H\varphi, \\ 0 < \left(1 - \frac{\mu}{\mu_0}\right)(\beta\theta^m + b\theta^{m-1}H\psi) \leq v_1 \leq v_0 \leq \beta\theta^m + b\theta^{m-1}H\psi. \end{cases}$$

By induction, we suppose that (3.3) and (3.4) hold for $k \in \mathbb{N}$.

Then since g is nondecreasing, we have

$$v_{k+2} - v_{k+1} = \mu V[q(g(u_k) - g(u_{k+1}))] \leq 0. \tag{3.5}$$

From the fact that f is nondecreasing and using inequality (3.5), we have

$$u_{k+2} - u_{k+1} = \lambda V[p(f(v_{k+1}) - f(v_{k+2}))] \geq 0. \tag{3.6}$$

Furthermore $v_k \geq 0$ implies that

$$u_{k+2} \leq \alpha\theta^m + a\theta^{m-1}H\varphi.$$

Taking into account the fact that g is nondecreasing and using (3.2) and (3.3), we get

$$\begin{aligned} v_{k+2} &= \beta\theta^m + b\theta^{m-1}H\psi - \mu V(qg(u_{k+1})) \geq \\ &\geq \beta\theta^m + b\theta^{m-1}H\psi - \mu V(qg(\alpha\theta^m + a\theta^{m-1}H\varphi)) \geq \\ &\geq \left(1 - \frac{\mu}{\mu_0}\right)(\beta\theta^m + b\theta^{m-1}H\psi). \end{aligned}$$

Hence (3.3) and (3.4) hold. Therefore, the sequences $(u_k)_{k\geq 0}$ and $(v_k)_{k\geq 0}$ converge respectively to two functions u and v satisfying

$$\begin{cases} 0 < \left(1 - \frac{\lambda}{\lambda_0}\right)(\alpha\theta^m + a\theta^{m-1}H\varphi) \leq u \leq \alpha\theta^m + a\theta^{m-1}H\varphi, \\ 0 < \left(1 - \frac{\mu}{\mu_0}\right)(\beta\theta^m + b\theta^{m-1}H\psi) \leq v \leq \beta\theta^m + b\theta^{m-1}H\psi. \end{cases} \tag{3.7}$$

Now we claim that

$$u = \alpha\theta^m + a\theta^{m-1}H\varphi - \lambda V(pf(v)) \tag{3.8}$$

and

$$v = \beta\theta^m + b\theta^{m-1}H\psi - \mu V(qg(u)). \tag{3.9}$$

It follows from the fact that f is nondecreasing and $H\psi$ is bounded, that for each $y \in \mathbb{R}^n_+$ and $k \in \mathbb{N}$

$$\begin{aligned} f(v_k(y))p(y) &\leq f(\beta y_n^m + b y_n^{m-1}H\psi(y))p(y) \leq \\ &\leq f(c y_n^{m-1}(y_n + 1))p(y) = y_n^{m-1}p_c(y). \end{aligned}$$

Moreover, since $p_c \in K^{\infty}_{m,n}$ we have by (2.1) that for each $x \in \mathbb{R}^n_+$

$$y \longmapsto G_{m,n}(x,y) y_n^{m-1} p_c(y) \in L^1(\mathbb{R}^n_+).$$

So using the continuity of f and the dominated convergence theorem we deduce that

$$\lim_{k\to\infty} V(pf(v_k)) = V(pf(v)).$$

This implies (3.8) by letting $k \to \infty$ in

$$u_k = \alpha\theta^m + a\theta^{m-1} H\varphi - \lambda V(pf(v_k)).$$

Similarly we have (3.9).

Next, we aim to prove that (u,v) satisfies (in the distributional sense)

$$\begin{cases} (-\Delta)^m u + \lambda f p(v) = 0 & \text{in} \quad \mathbb{R}^n_+, \\ (-\Delta)^m u + \mu q g(u) = 0 & \text{in} \quad \mathbb{R}^n_+. \end{cases}$$

From (3.8), we have obviously that

$$(-\Delta)^m u = -\lambda (-\Delta)^m V(pf(v)).$$

Now, combining (3.7) and the fact that f is nondecreasing, we get

$$V\left(pf(v)\right) \leq V\left(pf(c\theta^{m-1}(\theta+1))\right) = V\left(\theta^{m-1} p_c\right).$$

Since $q_c \in K^{\infty}_{m,n}$, then by Theorem 1.4 for $\beta = m-1$, we have

$$x \mapsto \frac{1}{x_n^{m-1}} V(\theta^{m-1} p_c)(x) \in C_0(\mathbb{R}^n_+). \tag{3.10}$$

We conclude due to Lemma 3.1 that

$$V(pf(v)) \in C(\mathbb{R}^n_+) \tag{3.11}$$

and consequently

$$V(pf(v)) \in L^1_{loc}(\mathbb{R}^n_+).$$

Hence $V\left(pf(v)\right)$ satisfies (in the distributional sense) the elliptic differential equation

$$(-\Delta)^m V\left(pf(v)\right) = pf(v) \quad \text{in} \quad \mathbb{R}^n_+.$$

It follows immediately from (3.8) and (3.11) that u is continuous. Similarly, we have

$$(-\Delta)^m V\left(qg(u)\right) = qg(u) \quad \text{in} \quad \mathbb{R}^n_+.$$

and v is continuous in $\mathbb{R}^n_+$.

Furthermore, since for $x \in \mathbb{R}^n_+$ we have

$$0 \leq \frac{V(pf(v))(x)}{x_n^{m-1}} \leq \frac{1}{x_n^{m-1}} V\left(\theta^{m-1} p_c\right)(x).$$

We deduce from (3.10) that

$$\lim_{x\to(\xi,0)} \frac{V(pf(v))(x)}{x_n^{m-1}} = 0, \quad \forall \xi \in \mathbb{R}^{n-1}.$$

Hence by (3.8) we obtain

$$\lim_{x\to(\xi,0)} \frac{u(x)}{x_n^{m-1}} = \lim_{x\to(\xi,0)} (\alpha x_n + aH\varphi(x)) = a\varphi(\xi), \quad \forall \xi \in \mathbb{R}^{n-1}.$$

Similarly

$$\lim_{x\to(\xi,0)} \frac{v(x)}{x_n^{m-1}} = b\psi(\xi), \quad \forall \xi \in \mathbb{R}^{n-1}.$$

On the other hand, we have for $x \in \mathbb{R}^n_+$

$$\frac{V(pf(v))(x)}{x_n^m} \le \frac{1}{x_n^m} V\left(\theta^{m-1} p_c\right)(x).$$

So, using (3.10), we get

$$\lim_{x_n\to\infty} \frac{V(pf(v))(x)}{x_n^m} = 0,$$

this yields

$$\lim_{x_n\to\infty} \frac{u(x)}{x_n^m} = \alpha.$$

By similar arguments, we obtain

$$\lim_{x_n\to\infty} \frac{v(x)}{x_n^m} = \beta.$$

The proof is complete. □

4. PROOF OF THEOREM 1.7

Proof of Theorem 1.7. Assume that $\lambda = \mu = a = b = 1$, α, $\beta \ge 0$ and the hypotheses (H_4) and (H_5) are satisfied. Let $\gamma = 1 + \alpha_{\widetilde{p}} + \alpha_{\widetilde{q}}$ where $\alpha_{\widetilde{p}}$ and $\alpha_{\widetilde{q}}$ are the constants defined in Proposition 2.1 (ii) and associated respectively to the functions $\widetilde{p}$ and $\widetilde{q}$ given in hypothesis (H_5).

We recall that Φ is a non-trivial nonnegative bounded continuous function on $\mathbb{R}^{n-1}$. Let us consider two nonnegative bounded continuous functions φ and ψ on $\mathbb{R}^{n-1}$ such that $\varphi \ge \gamma\Phi$ and $\psi \ge \gamma\Phi$.

It follows that for each $x \in \mathbb{R}^n_+$, we have

$$H\varphi(x) \ge \gamma H\Phi(x) \quad \text{and} \quad H\psi(x) \ge \gamma H\Phi(x). \tag{4.1}$$

We consider the non-empty closed convex set S given by

$$S = \{w \in C_b(\mathbb{R}^n_+) : H\Phi \le w \le H\varphi\}.$$

We define the operator T on S by

$$Tw = H\varphi - \frac{1}{\theta^{m-1}} V\left(pf\left[\beta\theta^m + \theta^{m-1}H\psi - V(qg(\alpha\theta^m + \theta^{m-1}w))\right]\right).$$

We aim to prove that T has a fixed point in S.

First we show that TS is relatively compact in $C_b(\mathbb{R}^n_+)$. Let $w \in S$, then since g is nonincreasing we deduce that

$$V(qg(\alpha\theta^m + \theta^{m-1}w)) \leq V(qg(\theta^{m-1}H\Phi)) = V(\widetilde{q}\theta^{m-1}H\Phi).$$

Which implies by (H_5) and (2.1) that

$$V(qg(\alpha\theta^m + \theta^{m-1}w)) \leq \alpha_{\widetilde{q}}\theta^{m-1}H\Phi. \tag{4.2}$$

This together with (4.1) yields

$$\begin{aligned}\beta\theta^m + \theta^{m-1}H\psi - V(qg(\alpha\theta^m + \theta^{m-1}w) &\geq \gamma\theta^{m-1}H\Phi - \alpha_{\widetilde{q}}\theta^{m-1}H\Phi = \\ &= (1+\alpha_{\widetilde{p}})\theta^{m-1}H\Phi \geq \theta^{m-1}H\Phi > 0.\end{aligned}$$

Hence, by the monotonicity of f, we get

$$pf\left(\beta\theta^m + \theta^{m-1}H\psi - V(qg(\alpha\theta^m + \theta^{m-1}w)\right) \leq pf(\theta^{m-1}H\Phi) = \theta^{m-1}H\Phi\widetilde{p}. \tag{4.3}$$

Since $H\Phi$ is bounded, we obtain

$$pf\left(\beta\theta^m + \theta^{m-1}H\psi - V(qg(\alpha\theta^m + \theta^{m-1}w)\right) \leq \|H\Phi\|_\infty \theta^{m-1}\widetilde{p},$$

which implies by using Proposition 2.5 for $\beta = m-1$, that the family of functions

$$\left\{\frac{1}{\theta^{m-1}} V\left[pf\left(\beta\theta^m + \theta^{m-1}H\psi - V(qg(\alpha\theta^m + \theta^{m-1}w))\right] : w \in S\right\}$$

is relatively compact in $C_0(\mathbb{R}^n_+)$ and since $H\varphi \in C_b(\mathbb{R}^n_+)$, we conclude that the family TS is relatively compact in $C_b(\mathbb{R}^n_+)$.

Next we prove that $TS \subset S$. Let $w \in S$, we have

$$T(w) \leq H\varphi.$$

Furthermore by (4.3) and (2.1) we obtain

$$V\left[pf\left(\beta\theta^m + \theta^{m-1}H\psi - V(qg(\alpha\theta^m + \theta^{m-1}w))\right] \leq V\left(\theta^{m-1}\widetilde{p}H\Phi\right) \leq \alpha_{\widetilde{p}}\theta^{m-1}H\Phi.$$

Then

$$T(w) \geq H\varphi - \alpha_{\widetilde{p}}H\Phi \geq (\gamma - \alpha_{\widetilde{p}})H\Phi \geq H\Phi.$$

Now, let us show the continuity of the operator T in S for the supremum norm. Let $(w_k)_{k\in\mathbb{N}}$ be a sequence in S which converges uniformly to a function w in S. Since g is nonincreasing we deduce that

$$qg(\alpha\theta^m + \theta^{m-1}w_k) \leq qg(\theta^{m-1}H\Phi) = \theta^{m-1}H\Phi\widetilde{q}.$$

Now, it follows from (H_5) and (2.1), that for each $x \in \mathbb{R}^n_+$,

$$y \longmapsto G_{m,n}(x,y)\theta^{m-1}(y)H\Phi(y)\widetilde{q}(y) \in L^1(\mathbb{R}^n_+).$$

We conclude by the continuity of g and the dominated convergence theorem that

$$\lim_{k\to\infty} V(qg(\alpha\theta^m + \theta^{m-1}w_k)) = V(qg(\alpha\theta^m + \theta^{m-1}w)) \tag{4.4}$$

and so from the continuity of f, we get

$$\lim_{k\to\infty} pf\left[\beta\theta^m + \theta^{m-1}H\psi - V(qg(\alpha\theta^m + \theta^{m-1}w_k))\right] =$$
$$= pf[\beta\theta^m + \theta^{m-1}H\psi - V(qg(\alpha\theta^m + \theta^{m-1}w))].$$

Using (4.3), for w_k, $k \in \mathbb{N}$, we obtain for each x, y in $\mathbb{R}^n_+$

$$G_{m,n}(x,y)p(y)f\left[\beta y_n^m + y_n^{m-1}H\psi(y) - V(qg(\alpha\theta^m + \theta^{m-1}w_k))(y)\right] \le$$
$$\le G_{m,n}(x,y)y_n^{m-1}H\Phi(y)\widetilde{p}(y).$$

Then combining (H_5) and (2.1), we get by the dominated convergence theorem that for each $x \in \mathbb{R}^n_+$,

$$Tw_k(x) \to Tw(x) \quad \text{as} \quad k \to +\infty.$$

Consequently, as TS is relatively compact in $C_b(\mathbb{R}^n_+)$, we deduce that the pointwise convergence implies the uniform convergence, namely,

$$\|Tw_k - Tw\|_\infty \to 0 \quad \text{as} \quad k \to +\infty.$$

Therefore, T is a continuous mapping from S to itself and so it is a compact mapping on S. Finally, the Schauder fixed-point theorem implies the existence of a function $w \in S$ such that $w = Tw$. We put for $x \in \mathbb{R}^n_+$

$$u(x) = \alpha x_n^m + x_n^{m-1}w(x), \tag{4.5}$$

and

$$v(x) = \beta x_n^m + x_n^{m-1}H\psi(x) - V(qg(u))(x). \tag{4.6}$$

Then

$$u(x) = \alpha x_n^m + x_n^{m-1}H\varphi(x) - V(pf(v))(x). \tag{4.7}$$

It remains to prove that (u, v) is a positive continuous solution of the problem (1.1) with $\lambda = \mu = a = b = 1$ and satisfying for each $x \in \mathbb{R}^n_+$

$$\alpha x_n^m + x_n^{m-1}H\Phi(x) \le u(x) \le \alpha x_n^m + x_n^{m-1}H\varphi(x) \tag{4.8}$$

and

$$\beta x_n^m + x_n^{m-1}H\Phi(x) \le v(x) \le \beta x_n^m + x_n^{m-1}H\psi(x). \tag{4.9}$$

Since $w \in S$, we have clearly from (4.5) that u satisfies (4.8).

On the other hand by (4.6), we have that for each $x \in \mathbb{R}^n_+$,

$$v(x) \leq \beta x_n^m + x_n^{m-1} H\psi(x).$$

Now, since g is nonincreasing and using that $u \geq \theta^{m-1} H\Phi$ we obtain

$$qg(u) \leq \theta^{m-1} \widetilde{q} H\Phi,$$

which implies by (H_5) and (2.1) that

$$V\left(qg(u)\right) \leq \alpha_{\widetilde{q}} \theta^{m-1} H\Phi.$$

So we get from (4.6)

$$v \geq \beta\theta^m + \theta^{m-1} H\psi - \alpha_{\widetilde{q}} \theta^{m-1} H\Phi,$$

which yields the claim (4.9) by using (4.1).

Using (4.7) we obtain

$$(-\Delta)^m u = -(-\Delta)^m V(pf(v)).$$

On the other hand, we have from (4.9) and the monotonicity of f that

$$pf(v) \leq \theta^{m-1} H\Phi \widetilde{p} \leq \|H\Phi\|_\infty \theta^{m-1} \widetilde{p},$$

which implies that

$$V\left(pf(v)\right) \leq V(\|H\Phi\|_\infty \theta^{m-1} \widetilde{p}). \tag{4.10}$$

Since we have from Theorem 1.4 that

$$x \mapsto \frac{1}{x_n^{m-1}} V(\|H\Phi\|_\infty \theta^{m-1} \widetilde{p})(x) \in C_0(\mathbb{R}^n_+), \tag{4.11}$$

we conclude due to Lemma 3.1 that

$$V(pf(v)) \in C(\mathbb{R}^n_+). \tag{4.12}$$

Therefore $V(pf(v)) \in L^1_{loc}(\mathbb{R}^n_+)$ and we have in the distributional sense that $(-\Delta)^m u = -pf(v)$. Next, combining (4.7) and (4.12) we get obviously that u is continuous.

Similarly, since $(-\Delta)^m v = -(-\Delta)^m V(qg(u))$, we obtain that

$$(-\Delta)^m v = -qg(u)$$

and v is continuous. Finally let $\xi \in \mathbb{R}^{n-1}$. From (4.10), we have for $x \in \mathbb{R}^n_+$

$$0 \leq \frac{V(pf(v))(x)}{x_n^{m-1}} \leq \frac{V(\|H\Phi\|_\infty \theta^{m-1} \widetilde{p})(x)}{x_n^{m-1}},$$

this yields by (4.11) that

$$\lim_{x \to (\xi,0)} \frac{V(pf(v))(x)}{x_n^{m-1}} = 0.$$

Thus by (4.7) we have

$$\lim_{x\to(\xi,0)} \frac{u(x)}{x_n^{m-1}} = \lim_{x\to(\xi,0)} \alpha x_n + H\varphi(x) - \frac{V(pf(v))(x)}{x_n^{m-1}} = \varphi(\xi).$$

Simirlary

$$\lim_{x\to(\xi,0)} \frac{v(x)}{x_n^{m-1}} = \lim_{x\to(\xi,0)} \beta x_n + H\psi(x) - \frac{V(qg(u))(x)}{x_n^{m-1}} = \psi(\xi).$$

Now, (4.10) and (4.11) imply that $\frac{V(pf(v))}{\theta^{m-1}}$ is bounded, so using (4.7) and taking into account that $H\varphi$ is also bounded we get

$$\lim_{x_n\to\infty} \frac{u(x)}{x_n^{m}} = \lim_{x_n\to\infty} \Big[\alpha + \frac{1}{x_n}\Big(H\varphi(x) - \frac{V(pf(v))(x)}{x_n^{m-1}}\Big)\Big] = \alpha.$$

Similarly, by (4.6) we have

$$\lim_{x_n\to\infty} \frac{v(x)}{x_n^{m}} = \lim_{x_n\to\infty} \Big[\beta + \frac{1}{x_n}\Big(H\psi(x) - \frac{V(qg(u))(x)}{x_n^{m-1}}\Big)\Big] = \beta.$$

This ends the proof. □

5. EXAMPLES

To illustrate Theorem 1.6, we give the following two examples.

Example 5.1. Let $\alpha = b = 1$ and $a = \beta = 0$. Let φ and ψ be two non-trivial nonnegative bounded continuous functions on $\mathbb{R}^{n-1}$ such that there exists $c_0 > 0$, satisfying $\psi(x) \geq c_0$ for all $x \in \mathbb{R}^{n-1}$.

We consider the functions $f,\, g : (0,\infty) \to [0,\infty)$ continuous and nondecreasing such that there exists $\eta > 0$ satisfying for each $t > 0$

$$0 \leq f(t) \leq \eta t \quad \text{and} \quad 0 \leq g(t) \leq \eta t.$$

We assume that p and q are nonnegative measurable functions on $\mathbb{R}^n_+$ such that

$$p_1 = \frac{p}{\theta}, \quad p_2 = (1+\theta)p, \quad q_1 = q\theta \quad \text{and} \quad q_2 = (1+\theta)q$$

are in $K^{\infty}_{m,n}$.

For each positive constant c, we have

$$p_c = \frac{p}{\theta^{m-1}} f(c\theta^{m-1}(\theta+1)) \leq \eta c p_2 \quad \text{and} \quad q_c = \frac{q}{\theta^{m-1}} g(c\theta^{m-1}(\theta+1)) \leq \eta c q_2.$$

So, it is clear that (H_1) and (H_2) are satisfied.

Moreover, we have

$$V(pf(\theta^{m-1}H\psi) \leq V(\eta\theta^{m-1}pH\psi) \leq \eta\|H\psi\|_\infty V(\theta^{m-1}p) \leq \eta\|H\psi\|_\infty V(p_1\theta^m).$$

Since $p_1 \in K^{\infty}_{m,n}$ and θ is harmonic in $\mathbb{R}^n_+$, we deduce by (2.1) that

$$V(pf(\theta^{m-1}H\psi)) \leq \eta \|H\psi\|_\infty \alpha_{p_1}\theta^m.$$

So for each $x \in \mathbb{R}^n_+$, we have

$$\frac{x_n^m}{V(pf(\theta^{m-1}H\psi)(x)} \geq \frac{x_n^m}{\eta\|H\psi\|_\infty \alpha_{p_1} x_n^m} \geq \frac{1}{\eta\|H\psi\|_\infty \alpha_{p_1}},$$

which implies that $\lambda_0 > 0$.

On the other hand, we have

$$V(qg(\theta^m)) \leq \eta V(q\theta^m) \leq \eta V(q_1\theta^{m-1}),$$

which implies by (2.1) that

$$V(qg(\theta^m)) \leq \eta \alpha_{q_1}\theta^{m-1}.$$

So, we obtain for $x \in \mathbb{R}^n_+$

$$\frac{x_n^{m-1}H\psi(x)}{V(qg(\theta^m))(x)} \geq \frac{c_0 x_n^{m-1}}{\eta\alpha_{q_1} x_n^{m-1}} \geq \frac{c_0}{\eta\alpha_{q_1}} > 0.$$

This proves that $\mu_0 > 0$. Hence (H_3) is satisfied.

Example 5.2. Let $m \geq 2$, $\alpha = b = 1$, $a = \beta = 0$ and φ, ψ be non-trivial nonnegative bounded continuous functions on $\mathbb{R}^{n-1}$. We consider f and g two continuous and nondecreasing functions on $(0,\infty)$ such that there exists $\eta > 0$ satisfying

$$0 \leq f(t) \leq \eta(1+t) \quad \text{and} \quad 0 \leq g(t) \leq \eta(1+t), \quad \forall t > 0.$$

We take p and q two nonnegative measurable functions in $\mathbb{R}^n_+$ satisfying for each $x \in \mathbb{R}^n_+$

$$p(x) \leq \frac{c}{(|x|+1)^{\mu-\lambda}x_n^\lambda} \qquad \text{with} \quad \lambda < m \qquad \text{and} \quad \mu > 2m+1,$$

and

$$q(x) \leq \frac{c}{(|x|+1)^{s-r}x_n^r} \qquad \text{with} \quad 0 \leq r < m \qquad \text{and} \quad s \geq 3m+n.$$

First, let $c > 0$ and $x \in \mathbb{R}^n_+$, we have

$$p_c(x) = \frac{p(x)}{(\theta(x))^{m-1}} f(c\theta^{m-1}(\theta+1))(x) \leq \frac{\eta}{(|x|+1)^{\mu-\lambda}x_n^{\lambda+m-1}} + \frac{\eta c}{(|x|+1)^{\mu-1-\lambda}x_n^\lambda}.$$

Since $\lambda < m+1 < \mu$ and $\lambda < 2m+1 < \mu$, we deduce by using Example 1.3, that $p_c \in K^{\infty}_{m,n}$. Similarly $q_c = \frac{q}{\theta^{m-1}} g(c\theta^{m-1}(\theta+1) \in K^{\infty}_{m,n}$.

Hence (H_1) and (H_2) are satisfied.

Next, observe that

$$V(pf(\theta^{m-1}H\psi)) \leq \eta V(p(\theta^{m-1}H\psi+1)) \leq \eta \|H\psi\|_\infty V(p\theta^{m-1}) + \eta V(p).$$

Using again Example 1.3, we have $p_1 = \frac{p}{\theta}$ and $p_0 = \frac{p}{\theta^m}$ are in $K^{\infty}_{m,n}$. Therefore, as in Example 5.1, we get

$$V(pf(\theta^{m-1}H\psi)) \leq \eta\left(\|H\psi\|_\infty \alpha_{p_1} + \alpha_{p_0}\right)\theta^m,$$

which implies for each $x \in \mathbb{R}^n_+$,

$$\frac{x_n^m}{V(pf(\theta^{m-1}H\psi))(x)} \geq \frac{x_n^m}{\eta\left(\|H\psi\|_\infty \alpha_{p_1} + \alpha_{p_0}\right)x_n^m} > 0.$$

This yields $\lambda_0 > 0$.

To show that $\mu_0 > 0$, we claim the following

$$V(qg(\theta^m))(x) \leq c\frac{x_n^m}{(|x|+1)^n}, \quad x \in \mathbb{R}^n_+. \tag{5.1}$$

Indeed, we have for each $x \in \mathbb{R}^n_+$

$$V(qg(\theta^m))(x) \leq \eta c \int_{\mathbb{R}^n_+} \frac{G_{m,n}(x,y)}{(1+|y|)^{s-r-m}y_n^r}dy. \tag{5.2}$$

To estimate the above integral, we consider $\gamma : \mathbb{R}^n_+ \to B$ the Möbius transformation defined by $\gamma(x) = e - \frac{2(x+e)}{|x+e|^2}$, where $e = (0,0,\ldots,0,1)$. Then a simple computation shows that for $x, y \in \mathbb{R}^n_+$, we have

$$G_{m,n}(x,y) = \left|\gamma'(x)\right|^{\frac{n-2m}{2}} \left|\gamma'(y)\right|^{\frac{n-2m}{2}} H_{m,n}(\gamma(x),\gamma(y)), \tag{5.3}$$

where $\left|\gamma'(x)\right| = \frac{2}{|x+e|^2}$ and $H_{m,n}$is the Green function of the operator $(-\Delta)^m$ on B with Dirichlet boundary conditions $u = \frac{\partial}{\partial\nu}u = \ldots = \frac{\partial^{m-1}}{\partial\nu^{m-1}}u = 0$ on $\partial B = \{x \in \mathbb{R}^n : |x| = 1\}$.

On the other hand, it is easy to see that

$$|x+e| \approx |x|+1, \quad x \in \mathbb{R}^n_+, \tag{5.4}$$

which implies that

$$\left|\gamma'(x)\right| \approx \frac{1}{(|x|+1)^2}, \quad x \in \mathbb{R}^n_+. \tag{5.5}$$

Since for $x \in \mathbb{R}^n_+$, we have $1 - |\gamma(x)|^2 = \frac{4x_n}{|x+e|^2}$, then by (5.4) we obtain

$$x_n \approx (1-|\gamma(x)|)(|x|+1)^2, \quad x \in \mathbb{R}^n_+. \tag{5.6}$$

Combining this with (5.3) and (5.5), we get for $x \in \mathbb{R}^n_+$

$$\int_{\mathbb{R}^n_+} \frac{G_{m,n}(x,y)}{(1+|y|)^{s-r-m} y_n^r} dy \leq \frac{c}{(1+|x|)^{n-2m}} \int_{\mathbb{R}^n_+} \frac{H_{m,n}(\gamma(x),\gamma(y))}{(1+|y|)^{s+r+n-3m}(1-|\gamma(y)|)^r} dy.$$

Put $z = \gamma(y)$, then we have $dy = \frac{2^n}{|z-e|^{2n}} dz$ and $1 + |y| \approx \frac{1}{|z-e|}$. Thus, for $x \in \mathbb{R}^n_+$, we arrive at

$$\int_{\mathbb{R}^n_+} \frac{G_{m,n}(x,y)}{(1+|y|)^{s-r-m} y_n^r} dy \leq \frac{c}{(1+|x|)^{n-2m}} \int_B \frac{H_{m,n}(\gamma(x),z)}{|z-e|^{n+3m-s-r}(1-|z|)^r} dz.$$

Using that $n + 3m - s - r \leq 0$, we have for $x \in \mathbb{R}^n_+$

$$\int_{\mathbb{R}^n_+} \frac{G_{m,n}(x,y)}{(1+|y|)^{s-r-m} y_n^r} dy \leq \frac{c}{(1+|x|)^{n-2m}} \int_B \frac{H_{m,n}(\gamma(x),z)}{(1-|z|)^r} dz.$$

Since $r < m$, then by [3, Proposition 3.10] and (5.6) we deduce that

$$\int_{\mathbb{R}^n_+} \frac{G_{m,n}(x,y)}{(1+|y|)^{s-r-m} y_n^r} dy \leq \frac{c}{(|x|+1)^{n-2m}} (1-|\gamma(x)|)^m \leq c\frac{x_n^m}{(|x|+1)^n},$$

which gives (5.1).

Finally taking into account that

$$H\psi(x) \geq c\frac{x_n}{(|x|+1)^n}, \quad x \in \mathbb{R}^n_+,$$

we get by (5.1), that for $x \in \mathbb{R}^n_+$

$$\frac{x_n^{m-1} H\psi(x)}{V(qg(\theta^m))(x)} \geq c > 0.$$

So $\mu_0 > 0$. Hence (H_3) is satisfied.

We end this section by an example as an application of Theorem 1.7.

Example 5.3. Let $\delta > 0$, $\eta > 0$, $f(t) = t^{-\delta}$ and $g(t) = t^{-\eta}$.

Let p and q be two nonnegative measurable functions on $\mathbb{R}^n_+$ such that

$$p(x) \leq \frac{c}{(|x|+1)^{\mu-\lambda} x_n^\lambda} \text{ with } \lambda < m(1-\delta) < \mu - n(1+\delta),$$

and

$$q(x) \leq \frac{c}{(|x|+1)^{r-s} x_n^s} \text{ with } s < m(1-\eta) < r - n(1+\eta).$$

Let Φ be a non-trivial nonnegative bounded continuous function on $\mathbb{R}^{n-1}$. Since for $x \in \mathbb{R}^n_+$ we have

$$H\Phi(x) \geq c\frac{x_n}{(|x|+1)^n}.$$

We obtain

$$\widetilde{p}(x) = p(x)\frac{f(\theta^{m-1}H\Phi)(x)}{(\theta(x))^{m-1}\,H\Phi(x)} \leq \frac{c}{(|x|+1)^{\mu-\lambda-n(1+\delta)}x_n^{\lambda+m(1+\delta)}}, \quad x \in \mathbb{R}^n_+.$$

Similarly

$$\widetilde{q}(x) \leq \frac{c}{(|x|+1)^{r-s-n(1+\eta)}x_n^{s+m(1+\eta)}}, \quad x \in \mathbb{R}^n_+.$$

Hence, by Example 1.3 we deduce that (H_4) is satisfied. So there exists a constant $\gamma = 1 + \alpha_{\widetilde{p}} + \alpha_{\widetilde{q}} > 1$ such that if φ and ψ are two nonnegative bounded continuous functions on $\mathbb{R}^{n-1}$ satisfying $\varphi \geq \gamma\Phi$ and $\psi \geq \gamma\Phi$ on $\mathbb{R}^{n-1}$, then for each $\alpha \geq 0$, $\beta \geq 0$, problem

$$\begin{cases} (-\Delta)^m u + pv^{-\delta} = 0 \text{ in } \mathbb{R}^n_+, \\ (-\Delta)^m v + qu^{-\eta} = 0 \text{ in } \mathbb{R}^n_+, \\ \lim_{x\to(\xi,0)} \frac{u(x)}{x_n^{m-1}} = \varphi(\xi), \quad \forall\, \xi \in \mathbb{R}^{n-1}, \\ \lim_{x_n\to+\infty} \frac{u(x)}{x_n^m} = \alpha, \\ \lim_{x\to(\xi,0)} \frac{v(x)}{x_n^{m-1}} = \psi(\xi), \quad \forall\, \xi \in \mathbb{R}^{n-1}, \\ \lim_{x_n\to+\infty} \frac{v(x)}{x_n^m} = \beta. \end{cases}$$

has a positive continuous solution (u, v) satisfying (1.7).

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