

WEYL'S THEOREM FOR ALGEBRAICALLY k -QUASICLASS A OPERATORS

Fugen Gao and Xiaochun Fang

Abstract. If T or T^* is an algebraically k -quasiclass A operator acting on an infinite dimensional separable Hilbert space and F is an operator commuting with T , and there exists a positive integer n such that F^n has a finite rank, then we prove that Weyl's theorem holds for $f(T) + F$ for every $f \in H(\sigma(T))$, where $H(\sigma(T))$ denotes the set of all analytic functions in a neighborhood of $\sigma(T)$. Moreover, if T^* is an algebraically k -quasiclass A operator, then α -Weyl's theorem holds for $f(T)$. Also, we prove that if T or T^* is an algebraically k -quasiclass A operator then both the Weyl spectrum and the approximate point spectrum of T obey the spectral mapping theorem for every $f \in H(\sigma(T))$.

Keywords: algebraically k -quasiclass A operator, Weyl's theorem, α -Weyl's theorem.

Mathematics Subject Classification: 47B20.

1. INTRODUCTION

We begin with some standard notation on Fredholm theory. Throughout this paper let \mathcal{H} be a separable complex Hilbert space with inner product $\langle \cdot, \cdot \rangle$. Let $B(\mathcal{H})$ and $K(\mathcal{H})$ denote respectively, the C^* -algebra of all bounded linear operators and the ideal of compact operators acting on \mathcal{H} . If $T \in B(\mathcal{H})$, we shall write $\ker T$ and $\text{ran} T$ for the null space and the range of T respectively. Also let $\alpha(T) = \dim \ker T$, $\beta(T) = \dim \ker T^*$ and let $\sigma(T)$, $\sigma_a(T)$ denote the spectrum, approximate point spectrum of T , respectively. Let $p = p(T)$ be the ascent of T ; i.e., the smallest nonnegative integer p such that $\ker T^p = \ker T^{p+1}$. If such an integer does not exist, we put $p(T) = \infty$. Analogously, let $q = q(T)$ be the descent of T ; i.e., the smallest nonnegative integer q such that $\text{ran} T^q = \text{ran} T^{q+1}$, and if such an integer does not exist, we put $q(T) = \infty$. It is well known that if $p(T)$ and $q(T)$ are both finite then $p(T) = q(T)$. Moreover, $0 < p(\lambda - T) = q(\lambda - T) < \infty$ precisely when λ is a pole of the resolvent of T , see Heuser [20, Proposition 50.2]. An operator $T \in B(\mathcal{H})$ is called Fredholm if $\text{ran} T$ is

closed and both $\ker T$ and $\mathcal{H}/\text{ran}T$ are finite dimensional. The index of a Fredholm operator $T \in B(\mathcal{H})$, denoted by $i(T)$, is given by the integer

$$i(T) = \alpha(T) - \beta(T).$$

An operator $T \in B(\mathcal{H})$ is called Weyl if it is Fredholm of index zero and Brown if it is Fredholm of finite ascent and descent. The essential spectrum $\sigma_e(T)$, the Weyl spectrum $\sigma_w(T)$ and the Browder spectrum $\sigma_b(T)$ of $T \in B(\mathcal{H})$ are defined by

$$\begin{aligned}\sigma_e(T) &= \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Fredholm}\}, \\ \sigma_w(T) &= \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Weyl}\}, \\ \sigma_b(T) &= \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Browder}\}.\end{aligned}$$

Let $\text{iso}\mathcal{K}$ denote the isolated points of $\mathcal{K} \subseteq \mathbb{C}$. We write

$$\begin{aligned}\pi_{00}(T) &= \{\lambda \in \text{iso}\sigma(T) : 0 < \alpha(T - \lambda) < \infty\}, \\ \pi_{00}^a(T) &= \{\lambda \in \text{iso}\sigma_a(T) : 0 < \alpha(T - \lambda) < \infty\},\end{aligned}$$

and

$$p_{00}(T) = \sigma(T) \setminus \sigma_b(T).$$

It is evident that $\sigma_e(T) \subseteq \sigma_w(T) \subseteq \sigma_b(T) = \sigma_e(T) \cup \text{acc}\sigma(T)$ and $p_{00}(T) \subseteq \pi_{00}(T) \subseteq \pi_{00}^a(T)$, where $\text{acc}\sigma(T) = \sigma(T) \setminus \text{iso}\sigma(T)$. It is well known that $\sigma_w(T)$ is non-empty and

$$\sigma_w(T) = \bigcap \{\sigma(T + K) : K \in K(\mathcal{H})\}.$$

It is interesting to note that $\sigma_b(T)$ can be characterized in a way parallel to the definition of $\sigma_w(T)$:

$$\sigma_b(T) = \bigcap \{\sigma(T + K) : K \in K(\mathcal{H}) \text{ and } KT = TK\}.$$

We say that Weyl's theorem holds for $T \in B(\mathcal{H})$ if

$$\sigma(T) \setminus \sigma_w(T) = \pi_{00}(T),$$

and that Browder's theorem holds for $T \in B(\mathcal{H})$ if

$$\sigma(T) \setminus \sigma_w(T) = p_{00}(T),$$

that is,

$$\sigma_w(T) = \sigma_b(T).$$

By definition,

$$\sigma_{ea}(T) = \bigcap \{\sigma_a(T + K) : K \in K(\mathcal{H})\}$$

is the essential approximate point spectrum, and

$$\sigma_{ab}(T) = \bigcap \{\sigma_a(T + K) : K \in K(\mathcal{H}) \text{ and } KT = TK\}$$

is the Browder approximate point spectrum. Let

$$\Phi_+(\mathcal{H}) = \{T \in B(\mathcal{H}) : \text{ran}T \text{ is closed and } \alpha(T) < \infty\},$$

and

$$\Phi_-(\mathcal{H}) = \{T \in \Phi_+(\mathcal{H}) : i(T) \leq 0\}.$$

In [29, Theorem 3.1], it was shown that

$$\sigma_{ea}(T) = \{\lambda \in \mathbb{C} : T - \lambda \notin \Phi_+(\mathcal{H})\}.$$

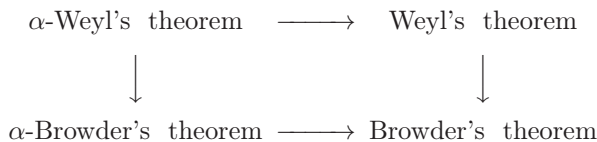
We say that α -Weyl's theorem holds for $T \in B(\mathcal{H})$ if

$$\sigma_\alpha(T) \setminus \sigma_{ea}(T) = \pi_{00}^a(T),$$

and that α -Browder's theorem holds for $T \in B(\mathcal{H})$ if

$$\sigma_{ea}(T) = \sigma_{ab}(T).$$

It is known [15,29] that if $T \in B(\mathcal{H})$ then we can express the implications between various Weyl's theorems and Browder's theorems in the following diagram.



H. Weyl [32] discovered that Weyl's theorem holds for hermitian operators and it has been extended from hermitian operators to hyponormal operators and to Toeplitz operators by L.A. Coburn [8], to cohyponormal operators by V. Rakočević [29], and to seminormal operators by S.K. Berberian [6,7]. And this result was generalized for p -hyponormal operators by M. Chō, M. Iton and S. Ōshiro [9], for class A operators by A. Uchiyama [31], for algebraically hyponormal operators by Y.M. Han [22] and for algebraically paranormal operators by R.E. Curto and Y.M. Han [10]. Recently, H. J. An and Y.M. Han [2] showed that Weyl's theorem holds for algebraically quasi-class A operators. Recall that $T \in B(\mathcal{H})$ is p -hyponormal for $p > 0$ if $(T^*T)^p - (TT^*)^p \geq 0$ [1]; when $p = 1$, T is called hyponormal. T is called cohyponormal, if T^* is hyponormal. If T is either hyponormal or cohyponormal, then T is called seminormal. And T is called paranormal if $\|Tx\|^2 \leq \|T^2x\|\|x\|$ for all $x \in \mathcal{H}$ [16,17]. In order to discuss the relations between paranormal and p -hyponormal and log-hyponormal operators (T is invertible and $\log T^*T \geq \log TT^*$), T. Furuta, M. Ito and T. Yamazaki [18] introduced a very interesting class of operators: class A defined by $|T^2| - |T|^2 \geq 0$, where $|T| = (T^*T)^{\frac{1}{2}}$ which is called the absolute value of T and they showed that class A is a subclass of paranormal operators and contains p -hyponormal and log-hyponormal ones. I.H. Jeon and I. H. Kim [23] introduced quasi-class A (i.e., $T^*(|T^2| - |T|^2)T \geq 0$) operators as an extension of the notion of class A operators. In this paper, we extend this result to algebraically k -quasiclass A operators using different methods.

Definition 1.1. $T \in B(\mathcal{H})$ is called a k -quasiclass A operator for a positive integer k if

$$T^{*k}(|T^2| - |T|^2)T^k \geq 0.$$

For interesting properties of k -quasiclass A operators, see [20,30]. In [30], this class of operators is called quasi-class (A, k) . We say that T is algebraically k -quasiclass A if there exists a nonconstant complex polynomial h such that $h(T)$ is k -quasiclass A.

Note that algebraically k -quasiclass A is preserved under translation by scalars and restriction to closed invariant subspaces.

In general, the following inclusions hold:

$$p\text{-hyponormal} \subseteq \text{class A} \subseteq \text{quasi-class A} \subseteq k\text{-quasiclass A} \subseteq \text{algebraically } k\text{-quasiclass A}.$$

Definition 1.2. An operator $T \in B(\mathcal{H})$ is said to have the single valued extension property (abbrev. SVEP) at $\lambda \in \mathbb{C}$ if for every open neighborhood \mathcal{G} of λ , the only function $f \in H(\mathcal{G})$ such that $(T - \mu)f(\mu) = 0$ on \mathcal{G} is $0 \in H(\mathcal{G})$, where $H(\mathcal{G})$ means the space of all analytic functions on \mathcal{G} having values in \mathcal{H} . Trivially, every operator $T \in B(\mathcal{H})$ has SVEP at points of the resolvent $\rho(T) = \mathbb{C} \setminus \sigma(T)$; moreover, from the identity theorem for analytic functions we have that every operator $T \in B(\mathcal{H})$ has SVEP at points of the boundary of $\sigma(T)$. In particular, every operator has SVEP at the isolated points of its spectrum. When T has SVEP at each $\lambda \in \mathbb{C}$, say that T has SVEP.

The single valued extension property dates back to the early days of local spectral theory and was introduced by N. Dunford [11,12]. This property plays a basic role in local spectral theory, see the recent monograph of K.B. Laursen and M.M. Neumann [26] or P. Aiena [3].

Definition 1.3. An operator $U \in B(\mathcal{H})$ is said to be a quasiaffinity if it is injective and has dense range. The operator $S \in B(\mathcal{H})$ is called a quasiaffine transform of $T \in B(\mathcal{H})$, notation $S \prec T$, if there exists a quasiaffinity $U \in B(\mathcal{H})$ such that $TU = US$. If both $S \prec T$ and $T \prec S$ hold, then S and T are called quasisimilar.

The quasinilpotent part $H_0(T - \lambda)$ and the analytic core $K(T - \lambda)$ of $T - \lambda$ are defined by

$$H_0(T - \lambda) = \{x \in \mathcal{H} : \lim_{n \rightarrow \infty} \|(T - \lambda)^n x\|^{\frac{1}{n}} = 0\}$$

and

$$K(T - \lambda) = \{x \in \mathcal{H} : \text{there exists a sequence } \{x_n\} \subseteq \mathcal{H} \text{ and } \delta > 0 \\ \text{for which } x = x_0, (T - \lambda)x_{n+1} = x_n \text{ and } \|x_n\| \leq \delta^n \|x\| \\ \text{for all } n = 1, 2, \dots\}.$$

We note that $H_0(T - \lambda)$ and $K(T - \lambda)$ are generally non-closed hyperinvariant subspaces of $T - \lambda$ such that $\ker(T - \lambda)^n \subseteq H_0(T - \lambda)$ for all $n = 0, 1, 2, \dots$ and $(T - \lambda)K(T - \lambda) = K(T - \lambda)$ [26].

Definition 1.4. An operator $T \in B(\mathcal{H})$ is said to be semi-regular if $\text{ran}T$ is closed and $\ker T \subseteq T^\infty(\mathcal{H}) = \bigcap_{n \in \mathbb{N}} \text{ran}T^n$.

Definition 1.5. An operator $T \in B(\mathcal{H})$ admits a generalized Kato decomposition, GKD for short, if there exists a pair of T -invariant closed subspaces (M, N) such that $\mathcal{H} = M \oplus N$, the restriction $T|_M$ is quasinilpotent and $T|_N$ is semi-regular. An operator $T \in B(\mathcal{H})$ has a GKD at every isolated point of $\sigma(T)$. We say that T is of Kato type at a point λ if $(T - \lambda)|_M$ is nilpotent in the GKD for $T - \lambda$.

See for details [3]. Recall that semi-Fredholm operators are of Kato type [24, Theorem 4]. For more information on semi-Fredholm operators, semi-regular operators and Kato type operators, see [3, 26].

2. WEYL'S THEOREM

FOR ALGEBRAICALLY k -QUASICLASS A OPERATORS

We start with the following lemmas which summarizes some basic properties of k -quasiclass A operators.

Lemma 2.1 ([20, 30]). *Let $T \in B(\mathcal{H})$ be a k -quasiclass A operator for a positive integer k and*

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \quad \text{on } \mathcal{H} = \overline{\text{ran}(T^k)} \oplus \ker T^{*k}$$

be 2×2 matrix expression. Assume that $\text{ran}(T^k)$ is not dense, then T_1 is a class A operator on $\overline{\text{ran}(T^k)}$ and $T_3^k = 0$. Furthermore, $\sigma(T) = \sigma(T_1) \cup \{0\}$.

Lemma 2.2 ([30]). *Let $T \in B(\mathcal{H})$ be a k -quasiclass A operator for a positive integer k and $\sigma(T) = \{\lambda\}$. Then $T = \lambda$ if $\lambda \neq 0$, and $T^{k+1} = 0$ if $\lambda = 0$.*

In [13], B.P. Duggal and S.V. Djordjević showed that quasinilpotent algebraically p -hyponormal operators are nilpotent. Recently, R.E. Curto and Y.M. Han in [10], II.J. An and Y.M. Han in [2] extended this result to algebraically paranormal operators and algebraically quasi-class A operators respectively. In the following theorem we show a similar result for algebraically k -quasiclass A operators.

Theorem 2.3. *Let $T \in B(\mathcal{H})$ be a quasinilpotent algebraically k -quasiclass A operator for a positive integer k . Then T is nilpotent.*

Proof. Let h be a complex nonconstant polynomial h such that $h(T)$ is k -quasiclass A. If $\text{ran}(h(T)^k)$ is dense, then $h(T)$ is a class A operator. Therefore T is an algebraically paranormal operator. We have that T is nilpotent by [10, 10, Lemma 2.2]. If $\text{ran}(h(T)^k)$ is not dense, then by Lemma 2.1 we can represent $h(T)$ as the upper triangular matrix

$$h(T) = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \quad \text{on } \mathcal{H} = \overline{\text{ran}(h(T)^k)} \oplus \ker h(T)^{*k},$$

where T_1 is a class A operator on $\overline{\text{ran}(h(T)^k)}$, $T_3^k = 0$ and $\sigma(h(T)) = \sigma(T_1) \cup \{0\}$. In fact, $\sigma(h(T)) = h(\sigma(T)) = \{h(0)\}$ for T is quasinilpotent. Since $\sigma(h(T)) = \sigma(T_1) \cup \{0\}$, we have $h(0) = 0$. Hence $h(T)$ is quasinilpotent. Since $h(T)$ is a

k -quasiclass A operator, by Lemma 2.2 $h(T)$ is nilpotent. Since $h(0) = 0$, we have $h(T) = \alpha T^m \prod_{i=1}^n (\lambda_i - T)$ for some natural number m and complex number α . Since $h(T)$ is nilpotent and all $\lambda_i - T$ are invertible, it follows that T is nilpotent. This completes the proof. \square

Recently, many excellent mathematicians have studied the Weyl type theorem of linear bounded operators defined on Banach spaces. Some of the basic results established for the Banach adjoint T' are also true for the T^* . For example, by means of the classical Fréchet-Riesz representation theorem we can deduce that the SVEP for T' and T^* are equivalent. In the following proofs, we shall apply this case.

An operator $T \in B(\mathcal{H})$ is said to be isoloid if every isolated point of $\sigma(T)$ is an eigenvalue of T and polaroid if every isolated point of $\sigma(T)$ is a pole of the resolvent of T . In general, if T is polaroid then it is isoloid. In [20], we proved that every k -quasiclass A operator is isoloid. Moreover we can prove the following result.

Theorem 2.4. *Let $T \in B(\mathcal{H})$ be an algebraically k -quasiclass A operator for a positive integer k . Then T has SVEP, and both T and T^* are polaroid. In particular, both T and T^* are isoloid.*

Proof. Let h be a nonconstant complex polynomial such that $h(T)$ is k -quasiclass A. By [20, Theorem 2.6], we know that $h(T)$ has finite ascent for all complex numbers. So we have that $h(T)$ has SVEP by [25, Proposition 1.8]. Hence T also has SVEP by [26, Theorem 3.3.9].

Now, if λ is an isolated point of $\sigma(T)$, $M = K(T - \lambda)$ and $N = H_0(T - \lambda)$, then (M, N) is a GKD for $T - \lambda$. Since $(T - \lambda)|_N$ is quasinilpotent and algebraically k -quasiclass A, it follows that $(T - \lambda)|_N$ is nilpotent by Theorem 2.3. Hence $T - \lambda$ is of Kato type. The SVEP for T and T^* at λ implies that both $p(T - \lambda)$ and $q(T - \lambda)$ are finite by [4, Theorem 2.3]. Hence λ is a pole of the resolvent of T .

Analogously, we shall prove that T^* is polaroid. Let λ be an isolated point of $\sigma(T^*)$. Then $\bar{\lambda}$ is an isolated point of $\sigma(T)$ and hence by the first part of the proof we have that $\bar{\lambda}$ is a pole of the resolvent of T . Therefore there exists a natural number n such that $n = p(T - \bar{\lambda}) = q(T - \bar{\lambda})$. Hence we have $\mathcal{H} = \ker(T - \bar{\lambda})^n \oplus \text{ran}(T - \bar{\lambda})^n$ and $\text{ran}(T - \bar{\lambda})^n$ is closed. From this we have $\mathcal{H} = (\ker(T - \bar{\lambda})^n)^\perp \oplus (\text{ran}(T - \bar{\lambda})^n)^\perp = \text{ran}(T^* - \lambda)^n \oplus \ker(T^* - \lambda)^n$. Hence $p(T^* - \lambda) = q(T^* - \lambda) < \infty$, that is, λ is a pole of the resolvent of T^* . This completes the proof. \square

Next we show that Weyl's theorem holds for $f(T)$, if T or T^* is an algebraically k -quasiclass A operator for a positive integer k .

Theorem 2.5. *Let T or T^* be an algebraically k -quasiclass A operator for a positive integer k . Then Weyl's theorem holds for $f(T)$ for every $f \in H(\sigma(T))$.*

Proof. Suppose that T is an algebraically k -quasiclass A operator for a positive integer k . We show first that Weyl's theorem holds for T . We use the fact [14, Theorem 2.2] that if T is polaroid then Weyl's theorem holds for T if and only if T has SVEP at points of $\lambda \notin \sigma_w(T)$. By Theorem 2.4, we have that T has SVEP and T is polaroid. Hence T satisfies Weyl's theorem.

We show that Weyl's theorem holds for $f(T)$. Since T is isoloid, by [27, Lemma] we have

$$\sigma(f(T)) \setminus \pi_{00}(f(T)) = f(\sigma(T) \setminus \pi_{00}(T)) = f(\sigma_w(T)),$$

where the last equality holds since T satisfies Weyl's theorem. Since T has SVEP, by [4, Corollary 2.6], we have $f(\sigma_w(T)) = \sigma_w(f(T))$. Therefore we have

$$\sigma(f(T)) \setminus \pi_{00}(f(T)) = \sigma_w(f(T)),$$

so Weyl's theorem holds for $f(T)$.

Suppose that T^* is an algebraically k -quasiclass A operator for a positive integer k . We show first that Weyl's theorem holds for T . We use the fact [4, Theorem 3.1] that if T or T^* has SVEP, then Weyl's theorem holds for T if and only if $\pi_{00}(T) = p_{00}(T)$. By Theorem 2.4, we have that T^* has SVEP. Hence it is sufficient to show that $\pi_{00}(T) = p_{00}(T)$. $p_{00}(T) \subseteq \pi_{00}(T)$ is clear, so we only need to prove $\pi_{00}(T) \subseteq p_{00}(T)$. Let $\lambda \in \pi_{00}(T)$. Then λ is an isolated point of $\sigma(T)$. Hence λ is a pole of the resolvent of T for T is polaroid by Theorem 2.4, that is, $p(\lambda - T) = q(\lambda - T) < \infty$. By assumption we have $\alpha(\lambda - T) < \infty$, so $\beta(\lambda - T) < \infty$. Hence we conclude that $\lambda \in p_{00}(T)$. Therefore Weyl's theorem holds for T . Since T^* has SVEP, by [4, Corollary 2.6], we have $f(\sigma_w(T)) = \sigma_w(f(T))$. Noting that T is isoloid, as in the proof of the first part, we have that Weyl's theorem holds for $f(T)$. This completes the proof. \square

Corollary 2.6. *Let T or T^* be an algebraically k -quasiclass A operator for a positive integer k . If F is an operator commuting with T and for which there exists a positive integer n such that F^n has a finite rank, then Weyl's theorem holds for $f(T) + F$ for every $f \in H(\sigma(T))$.*

Proof. Suppose T or T^* is an algebraically k -quasiclass A operator for a positive integer k . By Theorem 2.4 and Theorem 2.5, we have that T is isoloid and Weyl's theorem holds for $f(T)$. Observe that if T is isoloid then $f(T)$ is isoloid. The result follows from [28, Theorem 2.4]. \square

From the proof of Theorem 2.5, we have that the Weyl spectrum obeys the spectral mapping theorem for algebraically k -quasiclass A operators.

Corollary 2.7. *Let T or T^* be an algebraically k -quasiclass A operator for a positive integer k . Then for every $f \in H(\sigma(T))$, we have*

$$f(\sigma_w(T)) = \sigma_w(f(T)).$$

3. α -WEYL'S THEOREM

FOR ALGEBRAICALLY k -QUASICLASS A OPERATORS

For $T \in B(\mathcal{H})$, it is well known that $\sigma_{ea}(f(T)) \subseteq f(\sigma_{ea}(T))$ is always true for every $f \in H(\sigma(T))$. We have the following result that the essential approximate point spectrum obeys the spectral mapping theorem for k -quasiclass A operators.

Theorem 3.1. *Let T or T^* be an algebraically k -quasiclass A operator for a positive integer k . Then*

$$\sigma_{ea}(f(T)) = f(\sigma_{ea}(T))$$

for every $f \in H(\sigma(T))$.

Proof. We only need to prove that $f(\sigma_{ea}(T)) \subseteq \sigma_{ea}(f(T))$ since $\sigma_{ea}(f(T)) \subseteq f(\sigma_{ea}(T))$ is always true for any operator.

Suppose first that T is an algebraically k -quasiclass A operator for a positive integer k and let $f \in H(\sigma(T))$. Assume that $\lambda \notin \sigma_{ea}(f(T))$. Then we have $f(T) - \lambda \in \Phi_+(\mathcal{H})$ and $f(T) - \lambda = g(T) \prod_{i=1}^n (T - \lambda_i)$, where $\lambda_i \in \mathbb{C}$, $i = 1, 2, \dots, n$, and $g(T)$ is invertible.

Obviously, $\lambda \in f(\sigma_{ea}(T))$ if and only if $\lambda_i \in \sigma_{ea}(T)$ for some i . If $\lambda_i \notin \sigma_{ea}(T)$ for every $\lambda_i \in \{\lambda_i\}_{i=1}^n$, then we have $\lambda \notin f(\sigma_{ea}(T))$ and $f(\sigma_{ea}(T)) \subseteq \sigma_{ea}(f(T))$. In the following we shall prove that $\lambda_i \notin \sigma_{ea}(T)$ for every $\lambda_i \in \{\lambda_i\}_{i=1}^n$.

In fact, we have that every $T - \lambda_i \in \Phi_+(\mathcal{H})$ for $f(T) - \lambda \in \Phi_+(\mathcal{H})$. Since T is an algebraically k -quasiclass A operator, we have that T has SVEP by Theorem 2.4. Hence for each i we have $i(T - \lambda_i) \leq 0$ by [5, Theorem 2.6]. Therefore $T - \lambda_i \in \Phi_+(\mathcal{H})$, that is, $\lambda_i \notin \sigma_{ea}(T)$ for each i . As a consequence, $f(\sigma_{ea}(T)) \subseteq \sigma_{ea}(f(T))$.

Assume now that T^* is an algebraically k -quasiclass A operator. Then T^* has SVEP by Theorem 2.4. Then we have that $i(T - \lambda_i) \geq 0$ for each i by [5, Theorem 2.8]. Since

$$0 \leq \sum_{i=1}^n i(T - \lambda_i) = i(f(T) - \lambda) \leq 0,$$

we have $i(T - \lambda_i) = 0$ for each i . Since every $T - \lambda_i \in \Phi_+(\mathcal{H})$, we have $T - \lambda_i \in \Phi_+(\mathcal{H})$, that is, $\lambda_i \notin \sigma_{ea}(T)$ for each i . Hence $f(\sigma_{ea}(T)) \subseteq \sigma_{ea}(f(T))$. This completes the proof. \square

Theorem 3.2. *Let T^* be an algebraically k -quasiclass A operator for a positive integer k . Then α -Weyl's theorem holds for $f(T)$ for every $f \in H(\sigma(T))$.*

Proof. Suppose T^* is an algebraically k -quasiclass A operator for a positive integer k . We first prove that α -Weyl's theorem holds for T . We use the fact [4, Theorem 3.6] that if T^* has SVEP, then T satisfies α -Weyl's theorem if and only if T satisfies Weyl's theorem. Since T^* is an algebraically k -quasiclass A operator, we have that T^* has SVEP by Theorem 2.4 and T satisfies Weyl's theorem by Theorem 2.5. So T satisfies α -Weyl's theorem.

Next, we shall prove that α -Weyl's theorem holds for $f(T)$ for every $f \in H(\sigma(T))$. Let $f \in H(\sigma(T))$. Since T satisfies α -Weyl's theorem, we have that α -Browder's theorem holds for T . Hence $\sigma_{ea}(T) = \sigma_{ab}(T)$. Since T^* is an algebraically k -quasiclass A operator, it follows from Theorem 3.1 that

$$\sigma_{ea}(f(T)) = f(\sigma_{ea}(T)) = f(\sigma_{ab}(T)) = \sigma_{ab}(f(T)),$$

and so α -Browder's theorem holds for $f(T)$. We use the fact [15, Theorem 3.8] that if T satisfies α -Browder's theorem then α -Weyl's theorem holds for T if $\text{ran}(T - \lambda)$ is closed

for each $\lambda \in \pi_{00}^a(T)$. So it suffices to show that if $\lambda \in \pi_{00}^a(f(T))$, then $\text{ran}(f(T) - \lambda)$ is closed. Let $\lambda \in \pi_{00}^a(f(T))$. Then λ is an isolated point of $\sigma_a(f(T)) = f(\sigma_a(T))$ and $0 < \alpha(f(T) - \lambda) < \infty$. Since λ is an isolated point of $f(\sigma_a(T))$, if $\alpha_i \in \sigma_a(T)$, then α_i is an isolated point of $\sigma_a(T)$ by $f(T) - \lambda = g(T) \prod_{i=1}^n (T - \lambda_i)$ in Theorem 3.1. Since T^* has SVEP, we have that $\sigma_a(T) = \sigma(T)$ by [19, Corollary 7]. But T is isoloid by Theorem 2.4, so we have $0 < \alpha(T - \alpha_i) < \infty$ for each $i = 1, 2, \dots, n$. Since T satisfies α -Weyl's theorem, we have $\alpha_i \notin \sigma_{ea}(T)$ for each $i = 1, 2, \dots, n$. Hence $\text{ran}(f(T) - \lambda)$ is closed. This completes the proof. \square

Theorem 3.3. *Let T be an algebraically k -quasiclass A operator for a positive integer k and $S \prec T$. Then α -Browder's theorem holds for $f(S)$ for every $f \in H(\sigma(S))$.*

Proof. Since T is an algebraically k -quasiclass A operator for a positive integer k , we have that T has SVEP by Theorem 2.4. Hence α -Browder's theorem holds for $f(S)$ for every $f \in H(\sigma(S))$ by [10, Theorem 3.3]. This completes the proof. \square

Acknowledgements

We would like to express our gratitude to the referee for his or her useful suggestion. This work was supported by the National Natural Science Foundation of China (11071188) and the Natural Science Foundation of the Department of Education, Henan Province (2011A110009).

REFERENCES

- [1] A. Aluthge, *On p -hyponormal operators for $0 < p < 1$* , Integral Equations Operator Theory **13** (1990), 307–315.
- [2] H.J. An, Y.M. Han, *Weyl's theorem for algebraically quasi-class A operators*, Integral Equations Operator Theory **62** (2008), 1–10.
- [3] P. Aiena, *Fredholm and Local Spectral Theory with Applications to Multipliers*, Kluwer, 2004.
- [4] P. Aiena, *Classes of operators satisfying α -Weyl's theorem*, Studia Math. **169** (2005), 105–122.
- [5] P. Aiena, O. Monsalve, *Operators which do not have single valued extension property*, J. Math. Anal. **250** (2000), 435–448.
- [6] S.K. Berberian, *An extension of Weyl's theorem to a class of not necessarily normal operators*, Michigan Math. J. **16** (1969), 273–279.
- [7] S.K. Berberian, *The Weyl's Spectrum of an operator*, Indiana Univ. Math. J. **20** (1970), 529–544.
- [8] L.A. Coburn, *Weyl's theorem for non-normal operators*, Michigan Math. J. **13** (1966), 285–288.
- [9] M. Chō, M. Iton, S. Ōshiro, *Weyl's theorem holds for p -hyponormal operators*, Glasg. Math. J. **39** (1997), 217–220.

- [10] R.E. Curto, Y.M. Han, *Weyl's theorem for algebraically paranormal operators*, Integral Equations Operator Theory **47** (2003), 307–314.
- [11] N. Dunford, *Spectral theory I*, Pacific J. Math. **2** (1952), 559–614.
- [12] N. Dunford, *Spectral operator*, Pacific J. Math. **4** (1954), 321–354.
- [13] B.P. Duggal, S.V. Djordjević, *Weyl's theorem in the class of algebraically p -hyponormal operators*, Comment. Math. Prace Mat. **40** (2000), 49–56.
- [14] B.P. Duggal, *Polaroid operators satisfying Weyl's theorem*, Linear Algebra Appl. **414** (2006), 271–277.
- [15] S.V. Djordjević, Y. M. Han, *Browder's theorems and spectral continuity*, Glasg. Math. J. **42** (2000), 479–486.
- [16] T. Furuta, *On the class of paranormal operators*, Proc. Japan Acad. **43** (1967), 594–598.
- [17] T. Furuta, *Invitation to Linear Operators*, Taylor and Francis, London, 2001.
- [18] T. Furuta, M. Ito, T. Yamazaki, *A subclass of paranormal operators including class of log-hyponormal and several classes*, Sci. Math. **1** (1998) 3, 389–403.
- [19] J.K. Finch, *The single valued extension property on a Banach space*, Pacific J. Math. **58** (1975), 61–69.
- [20] F.G. Gao, X.C. Fang, *On k -quasiclass A operators*, J. Inequal. Appl. **2009** (2009), Article ID 921634, 1–10.
- [21] H. Heuser, *Functional Analysis*, John Wiley and Sons, 1982.
- [22] Y.M. Han, W.Y. Lee, *Weyl's theorem holds for algebraically hyponormal operators*, Proc. Amer. Math. Soc. **128** (2000), 2291–2296.
- [23] I.H. Jeon, I.H. Kim, *On operators satisfying $T^*|T^2|T \geq T^*|T|^2T$* , Linear Algebra Appl. **418** (2006), 854–862.
- [24] T. Kato, *Perturbation theory for nullity, deficiency and other quantities of linear operators*, J. Math. Anal. **6** (1958), 261–322.
- [25] K.B. Laursen, *Operators with finite ascent*, Pacific J. Math. **152** (1992), 323–336.
- [26] K.B. Laursen, M.M. Neumann, *Introduction to Local Spectral Theory*, Clarendon Press, Oxford, 2000.
- [27] W.Y. Lee, S.H. Lee, *A spectral mapping theorem for the Weyl spectral*, Glasg. Math. J. **38** (1996), 61–64.
- [28] M. Oudghiri, *Weyl's theorem and perturbations*, Integral Equations Operator Theory **53** (2005), 535–545.
- [29] V. Rakočević, *On the essential approximate point spectral II*, Mat. Vesnik **36** (1984), 89–97.
- [30] K. Tanahashi, I.H. Jeon, I.H. Kim, A. Uchiyama, *Quasinilpotent part of class A or (p, k) -quasihyponormal operators*, Oper. Theory Adv. Appl. **187** (2008), 199–210.
- [31] A. Uchiyama, *Weyl's theorem for class A operators*, Math. Inequal. Appl. **4** (2001), 143–150.

- [32] H. Weyl, *Über beschränkte quadratische Formen, deren differenz vollsteig ist*, Rend. Circ. Mat. Palermo **27** (1909), 373–392.

Fugen Gao
gaofugen08@126.com

Henan Normal University
College of Mathematics and Information Science
Xinxiang, Henan 453007, China

Tongji University
Department of Mathematics
Shanghai 200092, China

Xiaochun Fang
xfang@tongji.edu.cn

Department of Mathematics
Tongji University
Shanghai 200092, China

Received: October 5, 2010.

Revised: February 27, 2011.

Accepted: April 8, 2011.