INTEGRAL REPRESENTATION OF FUNCTIONS OF BOUNDED SECOND Φ-VARIATION IN THE SENSE OF SCHRAMM

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Abstract. In this article we introduce the concept of second Φ-variation in the sense of Schramm for normed-space valued functions defined on an interval $[a, b] \subset \mathbb{R}$. To that end we combine the notion of second variation due to de la Vallée Poussin and the concept of φ-variation in the sense of Schramm for real valued functions. In particular, when the normed space is complete we present a characterization of the functions of the introduced class by means of an integral representation. Indeed, we show that a function $f \in \mathbb{X}^{[a,b]}$ (where $\mathbb{X}$ is a reflexive Banach space) is of bounded second Φ-variation in the sense of Schramm if and only if it can be expressed as the Bochner integral of a function of (first) bounded variation in the sense of Schramm.

Keywords: Young function, Φ-variation, second Φ-variation of a function.

Mathematics Subject Classification: 26B30, 26B35.

1. INTRODUCTION

The concept of a function of bounded variation was introduced in 1881 by Camile Jordan ([10]) who carried out a rigorous study of the proof given by Dirichlet ([8]) on the convergence of the Fourier series of a function and exploited the fact that the concept was already implicit in the work of the latter. Ch.J. de la Vallée Poussin introduced in 1908 ([6]) the notion of second variation of a function. A few years later, in 1911, F. Riesz ([11]) proved that a function f is of bounded second variation on an interval $[a, b]$ if and only if it is the definite Lebesgue integral of a function f of bounded variation. Then in 1983 A.M. Russell and C.J.F. Upton ([12]) obtained a similar result for functions of bounded second variation in the sense of Wiener, showing that a function is of bounded second p-variation ($1 < p < \infty$) if and only if it is the definite Lebesgue integral of a function of bounded p-variation in the sense of Wiener. A common aspect of all mentioned results is that the maps considered are real valued functions. Recently (see [2]) these results were extended to the case of functions that take values in a Banach space $\mathbb{X}$. In this article we show that the Riesz's result also holds for the class of functions of bounded second variation in the sense of Schramm. More precisely, we will show that a function $f : [a,b] \to \mathbb{X}$, where $\mathbb{X}$ is a Banach space, is of second Φ-variation in the sense of Schramm ($f \in BV_{\Phi}^{2}([a,b],\mathbb{X})$) if and only if there exists a function $F : [a,b] \longrightarrow \mathbb{X}$ of bounded Φ-variation in the sense of Schramm ($F \in BV_{\Phi}([a,b],\mathbb{X})$) such that

$$f(t) = \int_a^t F(s)ds \quad \text{for all} \quad t \in [a,b].$$

The technics that we are going to use are similar to those applied by Russell and Upton in [12] and by Bracamonte, Giménez and Merentes in [2].

2. PRELIMINARIES

There are several equivalent definitions of the notion of functions of bounded variation. For the reader's convenience, in this section we present a summary account of some of the main results concerning the better known generalizations of the notion of functions of bounded variation.

Given an interval $[a,b] \subset \mathbb{R}$ and a function $f : [a,b] \to \mathbb{R}$. If $I = [c,d] \subset [a,b]$ we will use the following notations:

$$f[I] := f(d) - f(c),$$

$$f_2[I] := \frac{f(d) - f(c)}{d - c}.$$

By $\mathfrak{I}[a,b]$ we will denote the family of all sequences $\{I_n = [a_n,b_n]\}_{n\geq 0}$ of non-overlapping closed intervals contained in $[a,b]$ and such that $|I_n| := b_n - a_n > 0$ for all $n \geq 0$.

The notation $\pi[a,b]$ will be used for the set of all partitions $\xi = \{t_i\}_{i=1}^{n}$ of $[a,b]$, i.e., n is some positive integer and $a = t_0 < t_1 < \ldots < t_n = b$. When referring to such a partition ξ we will write $I_j = I_j(\xi) := [t_{j-1}, t_j]$.

The notation $\pi_3[a,b]$ will stand for the subset of $\pi[a,b]$ of all partitions containing at least three points.

Definition 2.1. A function $f : [a,b] \to \mathbb{R}$ is said to be of bounded variation on $[a,b]$ if there is a constant $M > 0$ such that

$$\sum_{n\geq 1} |f[I_n]| \leq M, \tag{2.1}$$

where $\{I_n\}_{n\geq 1}$ is any element of $\mathfrak{I}[a,b]$. The total variation of f on $[a,b]$ is denoted as $V(f;[a,b])$ or simply by $V(f)$, and it is the supremum of the sums (2.1) over $\mathfrak{I}[a,b]$.

It is readily seen that Definition 2.1 is equivalent to the following more familiar, textbook definition.

Definition 2.2. A function $f : [a, b] \longrightarrow \mathbb{R}$ is of bounded variation on $[a, b]$ if

$$V(f; [a, b]) := \sup_{\xi \in \pi[a,b]} \sum_{j=1}^{n} |f[I_j]| < \infty.$$

The class of all functions of bounded variation on $[a, b]$ is denoted as $BV[a, b]$.

The following results are well known.

Theorem 2.3 ([10])**.** *$f : [a, b] \longrightarrow \mathbb{R}$ is of bounded variation on $[a, b]$ if and only if it is the difference of two monotone functions.*

Theorem 2.4 ([3])**.** *$f : [a, b] \longrightarrow \mathbb{R}$ is of bounded variation on $[a, b]$ if and only if there is a non-decreasing function $\varphi : [a, b] \longrightarrow \mathbb{R}$ and a Lipschitz function $g : \varphi([a, b]) \longrightarrow \mathbb{R}$ with Lipschitz constant less or equal to one such that*

$$f(t) = (g \circ \varphi)(t), \quad t \in [a, b].$$

In 1937 N. Wiener ([14]) introduced the concept of functions of bounded p-variation ($1 < p < \infty$) as follows.

Definition 2.5 ([14])**.** A function $f : [a, b] \subset \mathbb{R} \to \mathbb{R}$ is said to be of bounded p-variation ($1 < p < \infty$) in the sense of Wiener iff

$$V_p^w(f; [a, b]) := \sup_{\xi \in \pi[a,b]} \sum_{j=1}^{n} |f[I_j]|^p < \infty.$$

The class of all functions of bounded p-variation on $[a, b]$, in the sense of Wiener, is denoted by $BV_p^w[a, b]$. Clearly, $BV_1^w[a, b] = BV[a, b]$. The relation

$$\|f\|_p := |f(a)| + (V_p^w(f; [a, b]))^{\frac{1}{p}}$$

defines a norm in $BV_p^w[a, b]$ with respect to which it becomes a Banach algebra.

For $f \in BV_p^w[a, b]$ and $t, s \in [a, b]$ let us define

$$\mathcal{V}(t) := V_p^w(f; [a, t]) \quad \text{and} \quad \upsilon(s) := V_p^w(f; [s, b]).$$

Proposition 2.6. *Suppose $f \in BV_p^w[a, b]$. Then:*

1. *If $t, s \in [a, b]$, then $|f(t) - f(s)|^p \le w(f; [a, b]) \le V_p^w(f; [a, b])$, where $w(f; [a, b]) := \sup\{d(f(s), f(t)) : t, s \in [a, b]\}$ is the so called modulus of continuity of f on $[a, b]$.*
2. *If $a \le t \le s \le b$, then:*
$$\mathcal{V}(t) \le \mathcal{V}(s),$$
$$\upsilon(s) \le \upsilon(t),$$
$$V_p^w(f; [t, s]) \le V_p^w(f; [a, b]) \quad (\textit{monotonicity}).$$
3. $\dfrac{V_p^w(f; [a, b])}{2^{p-1}} \le \mathcal{V}(s) + \upsilon(t) \le V_p^w(f; [a, b]).$

4. *If* $\varphi : [a, b] \to [c, d]$ *is a monotone function, then*

$$V_p^w(f; \varphi([a, b])) = V_p^w(f \circ \varphi; [a, b]).$$

5. $V_p^w(f; [a, b]) := \sup\{V_p^w(f; [t, s]) : \ t, s \in [a, b], \ t \leq s\}$.

The next proposition highlights the relation between the norm $\|\cdot\|_{BV_{[a,b]}}$ and the functional $V(\,\cdot\,; [a, b])$.

Proposition 2.7. *For* $f \in BV[a, b]$ *and* $c > 0$*, the estimate* $\|f\| \leq c$ *holds if and only if* $V(\frac{f}{c}) \leq 1$*. In particular,*

$$V\left(\frac{f}{\|f\|}; [a, b]\right) \leq 1 \tag{2.2}$$

for every $f \in BV[a, b]$ *with* $f(t) \not\equiv 0$.

The notion of bounded p-variation was extended by L.C. Young in [15]. The extension consisted in replacing the role played by the function $|t|^p$ $(1 < p < \infty)$ by a function in a more general class of convex functions, now known as Φ-functions.

Definition 2.8 (Φ-function)**.** A function $\varphi : [0, \infty) \to [0, \infty)$ is called a Φ-funcition if it satisfies the conditions:

1. φ is continuous on $[0, \infty)$,
2. $\varphi(t) = 0$ only if $t = 0$,
3. φ is non-decreasing,
4. $\varphi(t) \to \infty$ when $t \to \infty$.

If φ is a Φ-function, we will write $\varphi \in \Phi$.

Definition 2.9 (∞_1 condition)**.** A Φ-function φ is said to satisfy the condition ∞_1 if

$$\lim_{t\to\infty} \frac{\varphi(t)}{t} = \infty.$$

Definition 2.10. Let $\varphi \in \Phi$. A function $f : [a, b] \to \mathbb{R}$ is said to be of bounded φ-variation in the sense of Young if

$$V_\varphi(f; [a, b]) := \sup_{\xi\in\pi[a,b]} \sum_{j=1}^{n} \varphi\left(|f[I_j]|\right) < \infty.$$

The class of all functions of bounded φ-variation on $[a, b]$ in the sense of Young is denoted by $V_\varphi[a, b]$.

The following properties of the operator $V_\varphi(f; [x, y])$ are well known.

Proposition 2.11 ([4])**.** *Let* $f : [a, b] \to \mathbb{R}$ *be a function and let* $\varphi \in \Phi$*. Then:*

1. $\varphi(|f(t) - f(s)|) \leq w((f; [a, b])) \leq V_\varphi(f; [a, b])$ *for all* $s, t \in [a, b]$ *such that* $s < t$.
2. *If* $a \leq t \leq s \leq b$*, then* $V_\varphi(f; [a, t]) \leq V_\varphi(f; [a, s]) \leq V_\varphi(f; [s, a]) \leq V_\varphi(f; [t, a])$ *and* $V_\varphi(f; [t, s]) \leq V_\varphi(f; [a, b])$.

3. *If $t \in [a,b]$, then $V_\varphi(f;[a,t]) + V_\varphi(f;[t,b]) \le V_\varphi(f;[a,b])$.*
4. *If $\alpha : [a,b] \to [c,d]$ is a monotone function (not necessarily strict), then*

$$V_\varphi(f;[a,b]) = V_\varphi(f \circ \alpha;[a,b]).$$

5. *$V_\varphi(f;[a,b]) := \sup\{V_\varphi(f;[s,t]) :\ t,s \in [a,b]\}$.*

The class $V_\varphi[a,b]$ is not necessarily a linear space. However, imposing a natural condition on φ guarantees the desired linearity as shown in the following theorem.

Theorem 2.12 ([5])**.** *Let φ be a Φ-function. $V_\varphi([a,b])$ is a linear space if and only if φ satisfies a δ_2-condition, that is, there are constants t_0 and $k > 0$ such that*

$$\varphi(2t) \le k\varphi(t) \text{ for all } t \ge t_0.$$

On the other hand, $V_\varphi([a,b])$ is a symmetric, balanced and convex set and $V_\varphi(\,\cdot\,;[a,b])$ is a convex functional on it. Consequently, the linear space

$$BV_\varphi[a,b] := \{f : [a,b] \to \mathbb{R} \quad | \quad \exists\lambda > 0\colon V_\varphi(\lambda f;[a,b]) < \infty\}$$

can be equipped with a normed space structure by means of the norm:

$$\|f\|_\varphi := |f(a)| + \inf\left\{\lambda > 0 \mid V_\varphi\left(\frac{f}{\lambda};[a,b]\right) \le 1\right\}.$$

With this norm $BV_\varphi[a,b]$ actually becomes a Banach space.

As in the Wiener case the following proposition emphasizes the relation between $\|\cdot\|_\varphi$ and the functional $V_\varphi(\,\cdot\,;[a,b])$.

Proposition 2.13. *For $f \in BV_\varphi[a,b]$ and $c > 0$, the estimate $\|f\|_\varphi \le c$ holds if and only if $V_\varphi(\frac{f}{c}) \le 1$.*

In 1908 Charles Jean de la Vallée Poussin ([6]) introduced the notion of second variation of a real valued function defined on an interval $[a,b]$.

Definition 2.14. A function $f : [a,b] \longrightarrow \mathbb{R}$ is said to be of bounded second variation (and one writes $f \in BV^2[a,b]$) iff

$$V^2(f;[a,b]) := \sup_{\xi \in \pi_3[a,b]} \sum_{j=1}^{m-1} |f_2[I_{j+1}] - f_2[I_j]| < \infty.$$

With regard to this notion, the following facts are well known.

Theorem 2.15 ([6])**.** *$f \in BV^2[a,b]$ if and only if f can be expressed as the difference of two convex functions.*

Theorem 2.16 ([11])**.** *A function $f : [a,b] \to \mathbb{R}$ is of bounded second variation if and only if there is a function $F \in BV([a,b])$ such that*

$$f(x) = \int_a^x F(t)dt \text{ for all } x \in [a,b].$$

In 1983 A.M. Russell and C.J.F. Upton ([12]) introduced the class of real valued functions of bounded second variation on $[a,b]$, $BV_p^2[a,b]$, in the sense of Wiener, as follows.

Definition 2.17. $f \in BV_p^2[a,b]$ $(1 < p < \infty)$ iff

$$V_p^2(f;[a,b]) := \sup_{\xi \in \pi_3[a,b]} \sum_{j=0}^{n-2} |f_2[I_{j+2}] - f_2[I_{j+1}]|^p < \infty.$$

The following result ([12]) extends F. Riesz's theorem (Theorem 2.16) to the class $BV_p^2[a,b]$.

Theorem 2.18 ([12]). *$f : [a,b] \to \mathbb{R}$ is of bounded second p-variation in the sense of Wiener if and only if it is the definite integral of a function of bounded p-variation, in the sense of Wiener.*

Theorem 2.18 was extended recently (see [2]) to the case of functions of second bounded φ-variation in the sense of Young, where φ is a Φ-function that satisfies condition ∞_1.

Definition 2.19. Let φ be a Φ-function that satisfies condition ∞_1. A function $f : [a,b] \to \mathbb{R}$ is said to be of bounded second φ-variation in the sense of Young if

$$V_\varphi^2(f;[a,b]) := \sup_{\xi \in \pi_3[a,b]} \sum_{j=0}^{n-2} \varphi\left(|f_2[I_{j+2}] - f_2[I_{j+1}]|\right) < \infty.$$

Theorem 2.20 ([2]). *The function $f : [a,b] \to \mathbb{X}$, where $\mathbb{X}$ is a reflexive Banach space, is of bounded second φ-variation in the sense of Young if and only if it is the (Bochner) definite integral of a function of (first) bounded φ-variation in the sense of Young.*

3. SCHRAMM'S VARIATION

In the following lines we generalize the concept of variation given by Schramm ([13]) to functions defined on an interval $[a,b] \subset \mathbb{R}$ and that take values on a given normed space. To this end, we combine the Schramm's notion with the one of second variation due to de la Vallée Poussin in [6]. We also present some of the main properties of this class of functions.

Remember that by $\mathfrak{I}[a,b]$ we denote the family of all sequences $\{I_n = [a_n, b_n]\}_{n \geq 0}$ of non-overlapping closed intervals contained in $[a,b]$ and such that $|I_n| := b_n - a_n > 0$, for all $n \geq 0$.

We begin by recalling some of the main results and notations associated to the notion of bounded Φ-variation in the sense of Schramm.

Definition 3.1 (Φ-sequence)**.** A sequence of Φ-functions $\Phi = \{\varphi_n\}_{n\geq 1}$ is called a Φ-sequence if for all $t > 0$:

$$\varphi_{n+1}(t) \leq \varphi_n(t), \quad n \geq 1, \quad \text{and} \quad \sum_{n\geq 1} \varphi_n(t) = +\infty.$$

Definition 3.2. Let $\Phi = \{\varphi_n\}_{n\geq 1}$ be a Φ-sequence and $[a, b] \subset \mathbb{R}$ an interval. A function $f : [a, b] \to \mathbb{R}$ is said to be of bounded Φ-variation in the sense of Schramm if

$$V^s_{(\Phi,1)}(f; [a, b]) = V^s_{(\Phi,1)}(f) := \sup_{\{I_n\}\in\mathfrak{I}[a,b]} \sum_{n\geq 0} \varphi_n (|f[I_n]|) < \infty. \tag{3.1}$$

The class of all such functions is denoted by $V^S_{(\Phi,1)}[a, b]$. Notice that for $f \equiv$ *const.*, $V^S_{(\Phi,1)}(u; [a, b]) = 0$ and therefore $V^S_{(\Phi,1)}[a, b] \neq \emptyset$.

Remark 3.3. It is readily seen that in Definition 3.2 the $\sup_{\{I_n\}\in\mathfrak{I}[a,b]}$ can be replaced by the supremum over all finite collections $\{I_n\}^m_{n=1}$ in $\mathfrak{I}[a, b]$.

The following proposition summarizes some of the properties of this class of functions.

Theorem 3.4 ([13] or [9])**.** *Let $\Phi = \{\varphi_n\}_{n\geq 1}$ be a Φ-sequence. Then:*

1. $V^S_{(\Phi,1)}(f; [a, b]) = 0$ *if and only if* $f \equiv$ *const.*
2. $V^S_{(\Phi,1)}(f; [a, b]) < \infty \Rightarrow |f|_\infty \leq |f(a)| + \varphi_1^{-1}\left(V^S_{(\Phi,1)}(f)\right)$.
3. $V^S_{(\Phi,1)}[a, b]$ *is a symmetric and convex subset of* $\mathbb{R}^{[a,b]}$ *and* $V^S_{(\Phi,1)}(\,\cdot\,; [a, b])$ *is a convex functional on it.*
4. *The linear space* $BV^S_{(\Phi,1)}[a, b]$ *generated by* $V^S_{(\Phi,1)}[a, b]$ *is*

$$\left\{f : [a, b] \to \mathbb{R} \middle| \exists\lambda > 0\colon V^S_{(\Phi,1)}(\lambda f) < \infty\right\}.$$

5. $BV^S_{(\Phi,1)}[a, b]$ *is a Banach algebra with the norm*

$$\|f\|_{(\Phi,1)} = |f(a)| + \inf\left\{k > 0 : V^S_{(\Phi,1)}\left(\frac{f}{k}\right) \leq 1\right\}.$$

6. $\bigcup_\Phi BV^S_{(\Phi,1)}[a, b] = R[a, b]$[1)] *and* $\bigcap_\Phi BV^S_{(\Phi,1)}[a, b] = BV[a, b]$*, where both, unions and intersections, are taken over all Φ-sequences.*
7. $V^S_{(\Phi,1)}[a, b]$ *is a linear space if the sequence* $\Phi = \{\varphi_n\}_{n\geq 1}$ *satisfies a generalized* Δ_2*-condition; namely, for all* $t_0 > 0$ *there exists* $M(t_0) > 0$ *such that*

$$\sum_{n=1}^{m} \varphi_n(2t) \leq M(t_0) \sum_{n=1}^{m} \varphi_n(t) \text{ for all } t \geq t_0,\ m \geq 1.$$

[1)] The algebra of all functions in $\mathbb{R}^{[a,b]}$ that possess both one-sided limits at every point of (a, b).

8. *If $f \in BV^S_{(\Phi,1)}[a,b]$ and $c > 0$, the estimate $\|f\|^S_{(\Phi,1)} \le c$ holds if and only if $V^S_{(\Phi,1)}(\frac{f}{c}) \le 1$. In particular,*

$$V^S_B V^S_{(\Phi,1)}[a,b]\Big(\frac{f}{\|f\|^S_{(\Phi,1)}};[a,b]\Big) \le 1$$

for every $f \in BV^S_{(\Phi,1)}[a,b]$ with $f(t) \not\equiv 0$.

Now we present the mentioned extension.

Definition 3.5. Let $(\mathbb{X}, |\cdot|)$ be a normed space, let $\Phi = \{\varphi_n\}_{n\ge1}$ be a Φ-sequence and let $[a,b] \subset \mathbb{R}$ be an interval. A function $f : [a,b] \to \mathbb{X}$ is said to be of *bounded second Φ-variation in the sense of Schramm* if

$$V^s_{(\Phi,2)}(f;[a,b]) = V^s_{(\Phi,2)}(f) = \sup_{\{I_n\}\in\Im[a,b]} \sum_{n\ge0} \varphi_n\left(\left|f_2\left[I_{n+1}\right] - f_2\left[I_n\right]\right|\right) < \infty. \qquad (3.2)$$

The class of all the functions in $\mathbb{X}^{[a,b]}$ that satisfy (3.2) is not empty, for if $x, y \in \mathbb{X}$ are fixed and $f \equiv x$ or $f(t) := tx + y$ then $V^S_{(\Phi,1)}(f) = 0$. We will denote this class by $V^S_{(\Phi,2)}([a,b],\mathbb{X})$ or simply as $V^S_{(\Phi,2)}[a,b]$.

The next proposition shows some basic properties of the class $V^S_{(\Phi,2)}([a,b],\mathbb{X})$.

Proposition 3.6. *Let $\Phi = \{\varphi_n\}_{n\ge1}$ be a Φ-sequence and let $f : [a,b] \to X$ be a function. Then:*

1. *If $[c,d] \subset [a,b]$ and $V^s_{(\Phi,2)}(f;[a,b]) < \infty$, then $V^s_{(\Phi,2)}(f;[a,b]) < \infty$ and*

$$V^s_{(\Phi,2)}(f;[c,d]) \le V^s_{(\Phi,2)}(f;[a,b]).$$

2. *The functional $V^s_{(\Phi,2)} : V^S_{(\Phi,2)}[a,b] \to \mathbb{X}$, defined by*

$$V^s_{(\Phi,2)}(f) := V^s_{(\Phi,2)}(f;[a,b])$$

is convex.

3. *If λ is a complex number with $|\lambda| \le 1$, then $V^s_{(\Phi,2)}(\lambda f) \le |\lambda|\, V^s_{(\Phi,2)}(f)$.*

Proof. Part 1 follows readily from the definition. In order to prove parts 2 and 3 one uses the fact that each of the functions in Φ are convex functions. □

Definition 3.7 (Absolute continuity)**.** A mapping $f : [a,b] \longrightarrow \mathbb{X}$ is called absolutely continuous if there exists a function $\delta : (0,1) \longrightarrow (0,1)$ such that for any $\epsilon > 0$, any $n \in \mathbb{N}$ and any finite collection of points $\{a_i, b_i\}_{i=1}^n \subset [a,b]$ such that $a_1 < b_1 \le a_2 < b_2 \le a_3 < \ldots \le a_n < b_n$, the condition $\sum_{i=1}^n (b_1 - a_i) < \delta(\epsilon)$ implies $\sum_{i=1}^n |f(b_i) - f(a_i)| < \epsilon$.

4. MAIN RESULTS

In this section we present a generalization of Theorem 2.20 for functions of bounded second Φ-variation in the sense of Schramm. Indeed, we will prove the following result (see Corollary 4.6 below):

Let $\mathbb{X}$ be a reflexive Banach space and let $\Phi = \{\varphi_n\}_{n\geq 1}$ be a Φ-sequence. A function $f : [a,b] \to \mathbb{X}$ is of bounded second Φ-variation in the sense of Schramm if and only if it is the definite (Bochner) integral of a function of bounded Φ-variation in the sense of Schramm.

Throughout the rest of this work $\mathbb{X}$ will be assumed to be a Banach space.

Lemma 4.1. *Let $\Phi = \{\varphi_n\}_{n\geq 1}$ be Φ-sequence. If $f \in V^S_{(\Phi,2)}([a,b],\mathbb{X})$, then $f \in$ Lip $[a,b]$ and consequently f is absolutely continuous.*

Proof. Let $a \leq t_0 < t_1 < t_2 < t_2 \leq b$. Since φ_1 is non-decreasing and convex, by the definition of $V^S_{(\Phi,2)}(f;[a,b])$ we must have

$$\varphi_1\left(\frac{|f_2[I_3]-f_2[I_1]|}{2}\right) \leq \frac{1}{2}\varphi_1\left(|f_2[I_3]-f_2[I_2]|\right) + \frac{1}{2}\varphi_1\left(|f_2[I_2]-f_2[I_1]|\right) \leq \\ \leq V^S_{(\Phi,2)}(f;[a,b]),$$

where I_1, I_2, I_3 are non-overlapping intervals ($|I_j| > 0$) with end points in the set $\{a, t_0, t_1, t_2, t_3\}$. Fix a point $c \in (a,b)$ and consider any two other points $s,t \in [a,b]$. The proof will follow after analyzing the location of s,t with respect to a,b and c. We will use the notation $I_{xy} := [x,y]$.

Case 1. $a < s < c < t < b$. Then

$$\varphi_1\left(\frac{|f[I_{s,t}]|}{3}\right) \leq \frac{1}{3}\varphi_1\left(|f_2[I_{s,t}]-f_2[I_{t,b}]|\right) + \\ + \frac{1}{3}\varphi_1\left(|f_2[I_{t,b}]-f_2[I_{a,c}]|\right) + \frac{1}{3}\varphi_1\left(|f_2[I_{a,c}]|\right) \leq M',$$

where $M' := V^S_{(\Phi,2)}(f;[a,b]) + \varphi_1\left(|f_2[I_{a,c}]|\right)$.

Case 2. $a < s < c < t = b$. Then

$$\varphi_1\left(\frac{|f_2[I_{s,t}]|}{4}\right) \leq \frac{1}{4}\varphi_1\left(|f_2[I_{s,t}]-f_2[I_{a,s}]|\right) + \frac{1}{4}\varphi_1\left(|f_2[I_{a,s}]-f_2[I_{c,t}]|\right) + \\ + \frac{1}{4}\varphi_1\left(|f_2[I_{c,t}]-f_2[I_{a,c}]|\right) + \frac{1}{4}\varphi_1\left(|f_2[I_{a,c}]|\right) \leq M'.$$

Case 3. $a < s < t \leq c < b$. Then

$$\varphi_1\left(\frac{|f_2[I_{s,t}]|}{3}\right) \leq \frac{1}{3}\varphi_1\left(|f_2[I_{s,t}]-f_2[I_{c,b}]|\right) + \\ + \frac{1}{3}\varphi_1\left(|f_2[I_{c,b}]-f_2[I_{a,c}]|\right) + \frac{1}{3}\varphi_1\left(|f_2[I_{a,c}]|\right) \leq M'.$$

In the cases $a = s < c < t < b$, $a < c \leq s < t < b$ or $a = s < c < t = b$, we obtain

$$\varphi_1\left(\frac{|f_2[I_{s,t}]|}{4}\right) \leq M, \quad \text{where} \quad M := \max\{M', \varphi(|f_2[I_{a,b}]|)\}.$$

In any case we have

$$|f_2[I_{s,t}]| = \left|\frac{f(t)-f(s)}{t-s}\right| \leq \varphi^{-1}(4M).$$

Therefore, $f \in Lip\,[a,b]$. □

Remark 4.2. If $\mathbb{X}$ is a reflexive Banach space and $f \in V^S_{(\Phi,2)}([a,b],\mathbb{X})$ then the absolute continuity of f (Lemma 4.1) implies that f is strongly differentiable a.e. with derivative strongly measurable (see [1]).

In what follows the integral of a normed-space valued function defined on an interval $[a,b]$ means *the Bochner integral*. It is known that if a function is absolutely continuous then it is Bochner integrable on $[a,b]$ ([7]). By (the normed-space version of) property 2 of Theorem 3.4, any function in $V^S_{(\Phi,1)}([a,b],\mathbb{X})$ is Bochner integrable.

Theorem 4.3. *Let $\Phi = \{\varphi_n\}_{n\geq 1}$ be Φ-sequence. If $f \in V^S_{(\Phi,1)}([a,b],\mathbb{X})$, and we define $U(x) := \int_a^x f(t)dt$, then $U \in V^S_{(\Phi,2)}[a,b]$ and*

$$V^S_{(\Phi,2)}(U) \leq V^S_{(\Phi,1)}(f).$$

Proof. Let $\{I_n = [t_{n-1}, t_n]\}_{n\geq 1}$ be a sequence of intervals in $\mathfrak{I}([a,b])$. Then

$$\sum_{n\geq 1} \varphi_n(|U_2[I_{n+1}] - U_2[I_n]|) =$$

$$= \sum_{n\geq 1} \varphi_n\left(\left|\frac{1}{t_{n+1}-t_n}\int_{t_n}^{t_{n+1}} f(t)dt - \frac{1}{t_n - t_{n-1}}\int_{t_{n-1}}^{t_n} f(t)dt\right|\right) =$$

$$= \sum_{n\geq 1} \varphi_n\left(\left|\int_0^1 f(t_n + s(t_{n+1}-t_n))ds - \int_0^1 f(t_{n-1} + s(t_n - t_{n-1}))ds\right|\right)$$

and an application of Jensen inequality yields

$$\sum_{n\geq 1} \varphi_n(|U_2[I_{n+1}] - U_2[I_n]|) \leq$$

$$\leq \sum_{n\geq 1} \int_0^1 \varphi_n(|f(t_n + s(t_{n+1}-t_n)) - f(t_{n-1} + s(t_n - t_{n-1}))|)\,ds =$$

$$= \int_0^1 \sum \varphi(|f(t_n + s(t_{n+1}-t_n)) - f(t_{n-1} + s(t_n - t_{n-1}))|)\,ds \leq V^s_{(\Phi,1)}(f). \quad \square$$

Following the ideas of A.M. Russell and C.F. Upton in the proof of Lemma 6 of [12] and of M. Bracamonte, J. Giménez and N. Merentes (Lemma 3.2 of [2]), we get the next result.

Lemma 4.4. *Let $\Phi = \{\varphi_n\}_{n\geq 1}$ be Φ-sequence, E a dense subset of $[a,b]$ and let $f : E \to \mathbb{X}$ be a function such that there is a constant $K > 0$ with*

$$\sum_{k=0}^{n-1} \varphi_k\left(|f[I_j(\xi)]|\right) \leq K, \tag{4.1}$$

for any finite collection $\xi : a \leq t_0 < t_1 < \ldots < t_n \leq b$ in E. Then $g_E(x-0)$ exists for all $x \in (a,b]\backslash E$, where

$$g_E(x-0) := \lim_{\substack{h\to 0^+\\ x-h\in E}} g(x-h).$$

An analogous assertion holds for $g_E(x+0)$ $(x \in [a,b)\backslash E)$, which is similarly defined.

Proof. It suffices to show that $g(x-0)$ exists for all $t \in (a,b]\backslash E$. The case of $g_E(x+0)$ is treated analogously. We will proceed via proof by contradiction. Suppose that this is not the case, that is, suppose that there exists $x \in (a,b]\backslash E$ such that

$$\lim_{\substack{h\to 0^+\\ t-h\in E}} g(t-h) = \lim_{\substack{s\to t^-\\ s\in E}} g(s) \quad \text{does not exist.}$$

Let

$$\Lambda := \limsup_{\substack{x\to x_0^-\\ x\in E}} f(x) \quad \text{and} \quad \Gamma := \liminf_{\substack{x\to x_0^-\\ x\in E}} f(x).$$

Then $\Lambda > \Gamma$, and we can find two increasing sequences $\{x_n\}_{n\geq 0}$ and $\{y_n\}_{n\geq 0}$ such that

$$x_n < y_n < x_{n+1} < y_{n+1} < \ldots < x,$$

$$\lim_{n\to\infty} f(x_n) = \Lambda \quad \text{and} \quad \lim_{n\to\infty} f(y_n) = \Gamma.$$

If Λ and Γ are finite, consider $\varepsilon := \frac{\Lambda-\Gamma}{3}$ (otherwise take any $\varepsilon > 0$). Choose $n_\varepsilon \in \mathbb{N}$ such that

$$|f(x_n) - f(y_n)| > \varepsilon, \quad n > n_\varepsilon. \tag{4.2}$$

Since $\varphi_{n+1} \leq \varphi_n$, $n \geq 0$, (4.2) implies that for all $p > 0$

$$\sum_{k=n_\varepsilon+1}^{n_\varepsilon+p} \varphi_n\left(|f_n(x_n) - f_n(y_n)|\right) > \sum_{k=n_{\varepsilon+1}}^{n_\varepsilon+p} \varphi_{n_\varepsilon+p}(\varepsilon) > p\,\varphi_{n_\varepsilon+p}(\varepsilon),$$

which contradicts (4.1). □

Theorem 4.5. *Let $\mathbb{X}$ be a reflexive Banach space, $\Phi = \{\varphi_n\}_{n\geq 1}$ a Φ-sequence and suppose that $U \in V^S_{(\Phi,2)}([a,b],\mathbb{X})$. Then there exists a function $f \in V^S_{(\Phi,1)}[a,b]$ such that:*

(a) $U' = f$ *a.e.,*
(b) $U(x) := \int_a^x f(t)dt$,
(c) $V^S_{(\Phi,2)}(U) = V^S_{(\Phi,1)}(f)$.

Proof. Since F is absolutely continuous (Lemma 4.1) and $\mathbb{X}$ is a reflexive Banach space, f is strongly differentiable a.e., with derivative strongly measurable (see Remark 4.2). Let E be a set of zero Lebesgue measure such that F' exists at every point of the set $D := [a,b]\setminus E$. Given $m \in \mathbb{N}$, choose $m+1$ ordered points $a \leq x_0 < x_1 < \ldots < x_m \leq b$ in D. Now consider $m+2$ positive numbers: $h_0, h_1, \ldots, h_m$ and ξ such that $x_m - h_m, x_{m-1}+\xi, \ldots, x_k + h_k$, $k = 0, 1, \ldots, m-1$, are in D with

$$x_0 < x_0 + h_0 < x_1 < x_1 + h_1 < \ldots < x_{m-1} + h_{m-1} < x_{m-1} + \xi < x_m - h_m < x_m.$$

Then

$$\sum_{k=0}^{m-2} \varphi_k \left(\left| \frac{U(x_{k+1}+h_{k+1}) - U(x_{k+1})}{h_{k+1}} - \frac{U(x_k+h_k) - U(x_k)}{h_k} \right| \right) + \\ + \varphi_{m-1} \left(\left| \frac{U(x_m) - U(x_m - h_m)}{h_m} - \frac{U(x_{m-1}+\xi) - U(x_{m-1}+h_{m-1})}{\xi - h_{m-1}} \right| \right) \leq \\ \leq V^S_{(\Phi,2)}(U).$$

Taking the limits, in the above inequality, as $\xi \to 0$ and as $h_k \to 0$, $k = 0, \ldots, m$, we get

$$\sum_{k=0}^{m-1} \varphi_k \left(|U'(x_{k+1}) - U'(x_k)| \right) \leq V^S_{(\Phi,2)}(U). \tag{4.3}$$

If $a = x_0$ then we obtain $U'_+(a)$ instead of $U'(a)$ in (4.3). Thus, the derivative U' satisfies the conditions of Lemma 4.4. Now, let us define $f : [a,b] \to \mathbb{X}$, as

$$f(x) = \begin{cases} U'(x), & \text{when} \quad x \in D, \\ U'_D(x-0), & \text{when} \quad x \in (a,b] \setminus E, \\ U'_D(a+0), & \text{if} \quad x = a \notin D. \end{cases}$$

By construction, $U' = f$ a.e. By virtue of Theorem 4.3, we just need to verify that $f \in V^S_{(\Phi,1)}([a,b],\mathbb{X})$ and that $V^S_{(\Phi,1)}(f) \leq V^S_{(\Phi,2)}(U)$.

Let $A = \{I_k = [t_k, s_k]\}_{k=0}^m$ be any finite family of intervals in $\mathfrak{I}[a,b]$. We need to consider several cases.

Case 1. Suppose that there is just one $I_p \in A$ such that one of its end points is in E. Assume further that this end point is the right hand side one (s_p). Choose $s'_p \in D$ such that $t_p < s'_p < s_p$ and replace the interval I_p in A with $I'_p = [t_p, s'_p]$. Since all the end points of this new collection are in D and $f|_D = U$, we get

$$\sum_{k=0}^{p-2} \varphi_k\left(|f[I_{k+1}] - f[I_k]|\right) + \varphi_{p-1}\left(\left|f[I'_p] - f[I_{p-1}]\right|\right) + \\ + \varphi_p\left(\left|f[I_{p+1}] - f[I'_p]\right|\right) + \sum_{k=p+1}^{m-1} \varphi_k\left(|f[I_{k+1}] - f[I_k]|\right) \le V^S_{(\varphi,2)}(U).$$

Keeping s'_p in D and taking limit as $s'_p \to s_p$, we have $f(s'_p) \to f(s_p - 0)$. But in this case $f(s'_p) = U'(s'_p) \to U'(s_p - 0) = f(s_p)$. Thus

$$\sum_{k=0}^{m-1} \left(|f[I_{k+1}] - f[I_k]|\right) \le V^S_{(\varphi,2)}(U).$$

Case 2. If I_p is as in Case 1, but now t_p is the end point in E, then (since $A \in \mathfrak{I}[a,b]$ is finite) there is a point $t'_p \in D$, $t'_p < t_p$, such that $I'_p = [t'_p, s_p]$ does not overlap the rest of the intervals in A. Now we replace (in A) I_p with I'_p and proceed as in Case 1.
Case 3. Suppose now that just one point of E is a common end point of two intervals in A; say I_p and I_{p+1}. Then $t_p < s_p = t_{p+1} < s_{p+1}$. Choose $s'_p \in D$ such that $t_p < s'_p < s_p$ and replace I_p with $I'_p = [t_p, s'_p]$ and I_{p+1} with $I'_{p+1} = [s'_p, t_{p+1}]$. Since the end points of this new collection are in D we have

$$\sum_{k=0}^{p-2} \varphi_k\left(|f[I_{k+1}] - f[I_k]|\right) + \varphi_{p-1}\left(\left|f[I'_p] - f[I_{p-1}]\right|\right) + \varphi_p\left(\left|f[I'_{p+1}] - f[I'_p]\right|\right) + \\ + \varphi_{p+1}\left(\left|f[I_{p+2}] - f[I'_{p+1}]\right|\right) + \sum_{k=p+2}^{m-1} \varphi_k\left(|f[I_{k+1}] - f[I_k]|\right) \le V^S_{(\Phi,2)}(U).$$

Again, by considering the definition of f and passing to limit as $s'_p \to s_p$ (taking into account Lemma 4.4), one gets

$$\sum_{k=0}^{m-1} \left(|f[I_{k+1}] - f[I_k]|\right) \le V^S_{(\Phi,2)}(U).$$

Any other situation can be treated similarly. As claimed, we conclude that

$$f \in V^S_{(\Phi,1)}([a,b], \mathbb{X}) \quad \text{and} \quad V^S_{(\Phi,1)}(u) \le V^S_{(\Phi,2)}(U).$$

The proof is complete. □

The following result, which was already stated at the beginning of this section, now follows readily from Theorems 4.3 and 4.5.

Corollary 4.6. *Let $\mathbb{X}$ be a reflexive Banach space and let $\Phi = \{\varphi_n\}_{n\geq 1}$ be a Φ-sequence. A function $f : [a, b] \to \mathbb{X}$ is of bounded second Φ-variation in the sense of Schramm if and only if it is the definite (Bochner) integral of a function of bounded Φ-variation in the sense of Schramm.*

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