Opuscula Mathematica • Vol. 32 • No. 1 • 2012

A CHARACTERIZATION OF CONVEX φ-FUNCTIONS

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Abstract. The properties of four elements $(LPFE)$ and $(UPFE)$, introduced by Isac and Persson, have been recently examined in Hilbert spaces, L^p-spaces and modular spaces. In this paper we prove a new theorem showing that a modular of form $\rho_\Phi(f) = \int_\Omega \Phi(t, |f(t)|)\, d\mu(t)$ satisfies both $(LPFE)$ and $(UPFE)$ if and only if Φ is convex with respect to its second variable. A connection of this result with the study of projections and antiprojections onto latticially closed subsets of the modular space L^Φ is also discussed.

Keywords: inequalities, modulars, Orlicz-Musielak spaces, convexity, isotonicity, projections, antiprojections.

Mathematics Subject Classification: 39B62, 41A65, 46E30.

1. INTRODUCTION

"The (lower) property of four elements" $(LPFE)$ is a special inequality closely related to the isotonicity of projection operators. It was introduced in 1995 by Isac in [1], and later considered for Hilbert lattices, L^p-spaces and modular spaces in [2–4, 6].

In 1998, in [4] Isac and Persson began to examine another inequality, named "the upper property of four elements" $(UPFE)$, which is important for the study of antiprojection operators. This condition was also discussed in the paper [6].

In the present note we show that, under some not restrictive assumptions on the φ-function Φ, the Orlicz-Musielak modular generated by Φ satisfies both the properties of four elements if and only if the function Φ is convex with respect to its second variable (Theorem 3.5). This strengthens results from [2] and [6].

We also discuss a connection between our theorem and the study of projections and antiprojections onto latticially closed subsets of Orlicz-Musielak spaces. In particular, we present, as an open problem, another possible characterization of convexity (Problem 3.8).

2. PRELIMINARIES

In this section we have compiled some basic facts about modular spaces. For more details we refer the reader to [8].

From now on, $\mathbb{R}_+$ denotes the set of nonnegative real numbers and I_A stands for the characteristic map of a set A.

Definition 2.1. Let X be a linear space over the field $\mathbb{K}$. Then a function $\rho : X \to \mathbb{R}_+ \cup \{+\infty\}$ is called *a modular* iff the following properties are satisfied:

1. $\rho(x) = 0$ if and only if $x = 0$,
2. $\rho(\alpha x) = \rho(x)$ if $\alpha \in \mathbb{K}$ and $|\alpha| = 1$,
3. $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$ if $\alpha, \beta \geq 0$ and $\alpha + \beta = 1$.

The set $X_\rho = \left\{x \in X : \lim_{\alpha \to 0} \rho(\alpha x) = 0\right\}$, which is a linear subspace of X, will be called *the modular space* given by ρ.

We will always assume that the space X_ρ is ordered by some pointed convex cone $K \subset X_\rho$ in the following way: $x \geq y$ if and only if $x - y \in K$. We will also suppose that X_ρ is a lattice, i.e. the vectors $x \wedge y$ and $x \vee y$ exist for any $x, y \in X_\rho$, where $x \wedge y = \inf(x, y)$ and $x \vee y = \sup(x, y)$ (with respect to the ordering given by K).

Definition 2.2. A set $D \subset X_\rho$ will be called *latticially closed* iff $x \wedge y \in D$ and $x \vee y \in D$ for all $x, y \in D$.

Like in the theory of metric spaces, for any subset D of a modular space X_ρ we can define the projection and the antiprojection operators as below.

Definition 2.3. Choose any $x \in X_\rho$. Then we denote by $P_D(x)$ the set of all elements $y \in D$ such that

$$\rho(x - y) = \inf_{d \in D} \rho(x - d),$$

and we use the symbol $P^a_D(x)$ for the collection of all vectors $z \in D$ satisfying

$$\rho(x - z) = \sup_{d \in D} \rho(x - d).$$

The sets $P_D(x)$ and $P^a_D(x)$ are called *the projection of x onto D* and *the antiprojection of x onto D*, respectively.

Definition 2.4. We say that the projection operator P_D is *isotonic* iff, for any $x, y \in X_\rho$ such that $x \geq y$ and both $P_D(x)$, $P_D(y)$ are non-empty, there exist $w \in P_D(x)$ and $v \in P_D(y)$ satisfying $w \geq v$.

The antiprojection operator P^a_D is *antiisotonic* iff, for any $x, y \in X_\rho$ such that $x \geq y$ and both $P^a_D(x)$, $P^a_D(y)$ are non-empty, there exist $w \in P^a_D(x)$ and $v \in P^a_D(y)$ with $v \geq w$.

In this paper we restrict ourselves mainly to the case of Orlicz-Musielak spaces, defined in the following way.

Definition 2.5. Let (Ω, Σ, μ) be a space with a positive measure. Then $\Phi : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$ is *a φ-function with a parameter* iff it satisfies the following properties:

1. for every $t \in \Omega$, $\varphi_t(\cdot) = \Phi(t, \cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ is a nondecreasing, continuous function such that $\varphi_t(0) = 0$ and $\varphi_t(x) > 0$ for $x > 0$,
2. for every $x \in \mathbb{R}_+$, $\Phi(\cdot, x) : \Omega \to \mathbb{R}_+$ is a Σ-measurable function.

Definition 2.6. Let us now denote by X the set of all real Σ-measurable functions defined on Ω, with equality μ-almost everywhere. For $f \in X$, set

$$\rho_\Phi(f) = \int_\Omega \Phi(t, |f(t)|)\, d\mu(t).$$

Then ρ_Φ is a modular, which will be called *the Orlicz-Musielak modular* generated by Φ. The corresponding modular space $L^\Phi = X_{\rho_\Phi}$ is *the Orlicz-Musielak space*. We also define *the space of finite elements* of L^Φ:

$$E^\Phi = \{f \in L^\Phi : \rho_\Phi(\alpha f) < +\infty \text{ for all } \alpha > 0\}.$$

If the function Φ is independent of t, the modular ρ_Φ is said to be *the Orlicz modular*. Every Orlicz-Musielak space may be ordered by the cone $K = \{f \in L^\Phi : f \geq 0\}$.

Remark 2.7. Various examples of latticially closed subsets of Orlicz-Musielak spaces are given in [6, Ex. 2.4] and [7, Ex. 1.4].

3. MAIN RESULTS

We now recall the definition of the properties $(LPFE)$ and $(UPFE)$ in modular spaces.

Definition 3.1. We say that a modular ρ satisfies *the lower property of four elements* $(LPFE)$ iff, for any $x, y, w, z \in X_\rho$ such that $x \geq y$, we have

$$\rho(x - w) + \rho(y - z) \geq \rho(x - w \vee z) + \rho(y - w \wedge z),$$

and that it satisfies *the upper property of four elements* $(UPFE)$ iff, for any $x, y, w, z \in X_\rho$ such that $x \geq y$, we have

$$\rho(x - w) + \rho(y - z) \leq \rho(x - w \wedge z) + \rho(y - w \vee z).$$

In [2] and [6], the following theorem for Orlicz-Musielak modulars has been proved.

Theorem 3.2. *Suppose that Φ is a φ-function with a parameter which is convex with respect to its second variable. Then ρ_Φ has both the properties $(LPFE)$ and $(UPFE)$ (with respect to the cone of nonnegative functions).*

Let us now show two examples of non-convex Orlicz modulars which do not satisfy any of the properties of four elements.

Example 3.3. 1. Put $\Omega = [0,1]$ and $\Phi(t,x) = x^p$ with $p \in (0,1)$. Then, for $x = I_\Omega$, $y = 0$, $w = 2I_\Omega$ and $z = 3I_\Omega$, we have $x \geq y$, but

$$\rho_\Phi(x-w) + \rho_\Phi(y-z) = 1 + 3^p < 2^p + 2^p = \rho_\Phi(x - w \vee z) + \rho_\Phi(y - w \wedge z).$$

If $x = 2I_\Omega$, $y = I_\Omega$, $w = 4I_\Omega$ and $z = 3I_\Omega$, then

$$\rho_\Phi(x-w) + \rho_\Phi(y-z) = 2^p + 2^p > 1 + 3^p = \rho_\Phi(x - w \wedge z) + \rho_\Phi(y - w \vee z).$$

2. Let $\Omega = \mathbb{N}$ with the counting measure and define $\Phi(t,x) = \ln(x+1)$. In this case, for $x = I_{\{1\}}$, $y = 0$, $w = I_{\{1\}}$ and $z = 2I_{\{1\}}$, we get

$$\rho_\Phi(x-w) + \rho_\Phi(y-z) = \ln 3 < \ln 2 + \ln 2 = \rho_\Phi(x - w \vee z) + \rho_\Phi(y - w \wedge z).$$

Moreover, if $x = 2I_{\{1\}}$, $y = I_{\{1\}}$, $w = 3I_{\{1\}}$ and $z = 2I_{\{1\}}$, then

$$\rho_\Phi(x-w) + \rho_\Phi(y-z) = \ln 2 + \ln 2 > \ln 3 = \rho_\Phi(x - w \wedge z) + \rho_\Phi(y - w \vee z).$$

In fact, we will soon prove that Orlicz-Musielak modulars generated by a convex function are the only ones which satisfy $(LPFE)$ and $(UPFE)$. We will need the following additional assumption.

Assumption $(*)$ For any $y_1, y_2, y_3, y_4 \in \mathbb{R}_+$, $t \in \Omega$ and $\varepsilon > 0$, there exists $A^{y_i}_{t,\varepsilon} \in \Sigma$ such that:

(1) $\mu(A^{y_i}_{t,\varepsilon}) > 0$,
(2) $I_{A^{y_i}_{t,\varepsilon}} \in E^\Phi$,
(3) $|\Phi(\widetilde{t}, y_i) - \Phi(t, y_i)| < \varepsilon$ for $\widetilde{t} \in A^{y_i}_{t,\varepsilon}$ and $1 \leq i \leq 4$.

Remark 3.4. The condition $(*)$ holds in many important cases, some of which are listed below.

1. (the Orlicz case) Assume that $\Phi(t,x) = \Phi(x)$ and there exists $A \in \Sigma$ with $0 < \mu(A) < +\infty$. Then, for all y_1, y_2, y_3, y_4, t and ε, we may put $A^{y_i}_{t,\varepsilon} = A$.
2. (the sequence case) Let $\Omega = \mathbb{N}$ with the counting measure. Here we define $A^{y_i}_{t,\varepsilon} = \{t\}$.
3. (the continuous atomless case) Suppose that $\Omega \subset \mathbb{R}^n$ is an open set with positive Lebesgue measure and that, for each $x \in \mathbb{R}_+$, the function $\Phi(\cdot, x)$ is continuous. Then there exist positive numbers δ_i satisfying

$$|\Phi(\widetilde{t}, y_i) - \Phi(t, y_i)| < \varepsilon \quad \text{for} \quad \widetilde{t} \in B(t, \delta_i) \quad \text{and} \quad 1 \leq i \leq 4.$$

In this case, we set $A^{y_i}_{t,\varepsilon} = B(t,\delta)$ with $\delta = \min_i \delta_i$, where B denotes the closed ball in $\mathbb{R}^n$.

We may now formulate our main result, which strengthens Theorem 3.2.

Theorem 3.5. *Let $\Phi\colon \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$ be a φ-function with a parameter such that $(*)$ is satisfied. Then the following conditions are equivalent:*

(i) *For each $t \in \Omega$, φ_t is a convex function.*
(ii) *For each $t \in \Omega$, φ_t satisfies the Lim inequality: if $a, b \geq 0, c \geq a$, then*

$$\varphi_t(a) + \varphi_t(b + c) \geq \varphi_t(a + b) + \varphi_t(c).$$

(iii) *For each $t \in \Omega$, if $x_1, x_2, x_3, x_4 \in \mathbb{R}$ and $x_1 \geq x_3$, then*

$$\varphi_t(|x_1 - x_2|) + \varphi_t(|x_3 - x_4|) \geq \varphi_t(|x_1 - x_2 \vee x_4|) + \varphi_t(|x_3 - x_2 \wedge x_4|).$$

(iv) *For each $t \in \Omega$, if $x_1, x_2, x_3, x_4 \in \mathbb{R}$ and $x_1 \geq x_3$, then*

$$\varphi_t(|x_1 - x_2|) + \varphi_t(|x_3 - x_4|) \leq \varphi_t(|x_1 - x_2 \wedge x_4|) + \varphi_t(|x_3 - x_2 \vee x_4|).$$

(v) *ρ_Φ satisfies (LPFE).*
(vi) *ρ_Φ satisfies (UPFE).*

Proof. (i)⇒(ii) see [5, p. 195] and [2, p. 296–297].

(ii)⇒(iii)⇒(v) see [2, p. 297].

(ii)⇒(iv)⇒(vi) see [6, p. 602–603].

(v)⇒(iii) Suppose, on the contrary, that (iii) does not hold. Then, for some $t \in \Omega$, $\varepsilon > 0$ and $x_1, x_2, x_3, x_4 \in \mathbb{R}$ with $x_1 \geq x_3$,

$$\varphi_t(|x_1 - x_2|) + \varphi_t(|x_3 - x_4|) < \varphi_t(|x_1 - x_2 \vee x_4|) + \varphi_t(|x_3 - x_2 \wedge x_4|) - 4\varepsilon.$$

Applying $(*)$ to $y_1 = |x_1 - x_2|$, $y_2 = |x_3 - x_4|$, $y_3 = |x_1 - x_2 \vee x_4|$ and $y_4 = |x_3 - x_2 \wedge x_4|$, we get

$$\varphi_{\tilde{t}}(|x_1 - x_2|) + \varphi_{\tilde{t}}(|x_3 - x_4|) < \varphi_{\tilde{t}}(|x_1 - x_2 \vee x_4|) + \varphi_{\tilde{t}}(|x_3 - x_2 \wedge x_4|)$$

for all $\tilde{t} \in C = A^{y_i}_{t,\varepsilon}$. Set $x = x_1 \cdot I_C$, $y = x_3 \cdot I_C$, $w = x_2 \cdot I_C$, $z = x_4 \cdot I_C$. Integrating the last inequality over C and using $(*)$ again, we obtain

$$\rho_\Phi(x - w) + \rho_\Phi(y - z) < \rho_\Phi(x - w \vee z) + \rho_\Phi(y - w \wedge z),$$

which contradicts (v).

(vi)⇒(iv) This follows by the same method as in the previous part.

(iii)⇒(ii) Fix $t \in \Omega$ and $a, b, c \geq 0$ such that $c \geq a$. Define $x_1 = b + c$, $x_2 = 0$, $x_3 = c$, $x_4 = c - a$. Then $x_1 \geq x_3$ and, by (iii),

$$\begin{aligned}\varphi_t(b + c) + \varphi_t(a) &= \varphi_t(|x_1 - x_2|) + \varphi_t(|x_3 - x_4|) \geq \\ &\geq \varphi_t(|x_1 - x_2 \vee x_4|) + \varphi_t(|x_3 - x_2 \wedge x_4|) = \\ &= \varphi_t(a + b) + \varphi_t(c).\end{aligned}$$

(iv)⇒(ii) Choose t, a, b, c as above and put $x_1 = b + c$, $x_2 = c - a$, $x_3 = c$, $x_4 = 0$. Then, according to (iv), we have

$$\begin{aligned}\varphi_t(a + b) + \varphi_t(c) &= \varphi_t(|x_1 - x_2|) + \varphi_t(|x_3 - x_4|) \leq \\ &\leq \varphi_t(|x_1 - x_2 \wedge x_4|) + \varphi_t(|x_3 - x_2 \vee x_4|) = \\ &= \varphi_t(b + c) + \varphi_t(a).\end{aligned}$$

(ii)⇒(i) Fix $0 \le x \le y$ and set $a = x$, $b = \frac{y-x}{2}$, $c = \frac{x+y}{2}$. Then for all $t \in \Omega$, by the Lim inequality,

$$\varphi_t(x) + \varphi_t(y) = \varphi_t(a) + \varphi_t(b + c) \ge \varphi_t(a + b) + \varphi_t(c) = 2\varphi_t\left(\frac{x+y}{2}\right).$$

Consequently, φ_t is a continuous Jensen-convex function, so it is convex. □

It has been proved that $(LPFE)$ implies the isotonicity of every projection operator onto latticially closed set (see [2, Th. 2]), and $(UPFE)$ gives the antiisotonicity of all antiprojection operators onto such sets (see [6, Th. 3.10]). In consequence, by Theorem 3.2 we have

Theorem 3.6. *If Φ is convex with respect to its second variable, then:*

(a) *For each latticially closed set $D \subset L^\Phi$, the projection operator P_D is isotonic.*
(b) *For each latticially closed set $D \subset L^\Phi$, the antiprojection operator P_D^a is antiisotonic.*

Our last example shows that the conditions (a) and (b) may not be satisfied if we drop the assumption of convexity (i.e. if the properties of four elements do not hold).

Example 3.7. 1. Let $\Omega = [0, 2]$ and $\Phi(t, x) = x^p$ with $p \in (0, 1)$ such that $3^p - 2^p < \frac{1}{3}$. Define the latticially closed set

$$D = \left\{\alpha I_\Omega : 1 \ge \alpha \ge \frac{3}{4}\right\} \subset L^\Phi$$

and put $\Omega_1 = [0, \frac{1}{2})$, $\Omega_2 = [\frac{1}{2}, 2]$, $\varepsilon_1 = \frac{1}{4}$, $\varepsilon_2 = \frac{3}{4}$, $x_i = I_{\Omega_1} + \varepsilon_i I_{\Omega_2}$ for $i = 1, 2$.

Then, for $d = \alpha I_\Omega \in D$, we get

$$\rho_\Phi(x_i - d) = \frac{1}{2}(1 - \alpha)^p + \frac{3}{2}(\alpha - \varepsilon_i)^p = g_i(\alpha).$$

It is easy to calculate that g_i has its maximum at $\alpha_i = \dfrac{1 + \varepsilon_i \cdot \beta}{1 + \beta} \in (\frac{3}{4}, 1)$ with $\beta = 3^{\frac{1}{p-1}}$. Therefore $P_D(x_i) \subset \left\{\frac{3}{4}I_\Omega, I_\Omega\right\}$. Moreover, by the choice of p,

$$\rho_\Phi\left(x_1 - \frac{3}{4}I_\Omega\right) = \frac{1}{2}\left(\frac{1}{4}\right)^p + \frac{3}{2}\left(\frac{1}{2}\right)^p > \frac{3}{2}\left(\frac{3}{4}\right)^p = \rho_\Phi(x_1 - I_\Omega)$$

and

$$\rho_\Phi\left(x_2 - \frac{3}{4}I_\Omega\right) = \frac{1}{2}\left(\frac{1}{4}\right)^p < \frac{3}{2}\left(\frac{1}{4}\right)^p = \rho_\Phi(x_2 - I_\Omega).$$

Hence $P_D(x_1) = \{I_\Omega\}$ and $P_D(x_2) = \{\frac{3}{4}I_\Omega\}$. As $x_2 \ge x_1$, we see that P_D is not isotonic.

2. Choose any $p \in (0, 1)$ and define Ω, Ω_1, Ω_2, Φ as above. Fix two real numbers $\varepsilon_1, \varepsilon_2$ such that $0 \le \varepsilon_1 < \varepsilon_2 < 1$, $\dfrac{\varepsilon_2 - \varepsilon_1}{1 - \varepsilon_2} \le \beta = 3^{\frac{1}{p-1}}$ and set $x_i = \varepsilon_i I_{\Omega_1} + I_{\Omega_2}$, $i = 1, 2$.

Then $x_2 \geq x_1$, and for $d \in D = \{\alpha I_\Omega : 1 \geq \alpha \geq \varepsilon_2\}$ we obtain

$$\rho_\Phi(x_i - d) = \frac{1}{2}(\alpha - \varepsilon_i)^p + \frac{3}{2}(1-\alpha)^p.$$

Using the differential calculus again, we get

$$P_D^a(x_i) = \{\alpha_i I_\Omega\} \text{ with } \alpha_i = \frac{\varepsilon_i + \beta}{1+\beta} \in (\varepsilon_2, 1).$$

Since $\alpha_2 > \alpha_1$, the operator P_D^a is not antiisotonic.

Theorems 3.5, 3.6 and Example 3.7 suggest the following final question.

Problem 3.8. Is the convexity of Φ not only a sufficient, but also the necessary condition for (a) and (b)? In other words, do all Orlicz-Musielak modulars which do not satisfy $(LPFE)$ [resp. $(UPFE)$], also fail to have the property (a) [resp. (b)]?

If the answer was positive, we would get another interesting characterization of convexity. For us, however, it is still an open problem.

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Received: December 1, 2010.
Accepted: June 10, 2011.