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UNIFORMLY CONTINUOUS COMPOSITION OPERATORS IN THE SPACE OF BOUNDED Φ-VARIATION FUNCTIONS IN THE SCHRAMM SENSE

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Abstract. We prove that any uniformly continuous Nemytskii composition operator in the space of functions of bounded generalized Φ-variation in the Schramm sense is affine. A composition operator is locally defined. We show that every locally defined operator mapping the space of continuous functions of bounded (in the sense of Jordan) variation into the space of continous monotonic functions is constant.

Keywords: Φ-variation in the sense of Schramm, uniformly continuous operator, regularization, composition operator, Jensen equation, locally defined operators.

Mathematics Subject Classification: 47H30.

1. INTRODUCTION

Let $I = [a, b]$ $(a, b \in \mathbb{R}, a < b)$ be an interval of $\mathbb{R}$, X a real normed space, C a closed convex subset of X, Y a real Banach space and $h : I \times C \to Y$. Denote by C^I (or Y^I) the family of all functions $f : I \longrightarrow C$ (or $f : I \to Y$) and by $H : C^I \longrightarrow Y^I$ the Nemytskii composition operator generated by the function h defined by

$$H(f)(t) = h(t, f(t)), \quad t \in I, \quad f \in C^I. \tag{1.1}$$

Let $(\Phi BV(I, X), \|\cdot\|_\Phi)$ be a Banach space of functions $f : I \to X$ which are of bounded Φ -variation in the sense of Schramm, where the norm $\|\cdot\|_\Phi$ is defined with the aid of the Luxemburg-Nakano-Orlicz seminorm [8, 14, 16].

Assume that H maps the set of functions $f \in \Phi BV(I, X)$ such that $f(I) \subset C$ into $\Phi BV(I, Y)$. In the present paper we prove that, if H is uniformly continuous, then the left and right regularization of its generator h with respect to the first variable are affine functions in the second variable. This extends the main result of paper [4].

2. PRELIMINARIES

In this section we recall some facts which will be needed further on.

Denote by $\mathbb{R}$ the set of all real numbers and put $\mathbb{R}_+ = [0, \infty)$. We will say that a function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a φ-function if φ is continuous on $\mathbb{R}_+$, $\varphi(0) = 0$, φ is increasing on $\mathbb{R}_+$ and $\varphi(t) \to \infty$ as $t \to \infty$.

Let $\mathcal{F}$ be the set of all convex functions $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ such that $\varphi(0) = 0, \varphi(t) > 0$ for $t > 0$. Next, let $\Phi = \{\phi_n\}$ be a sequence of functions from $\mathcal{F}$ such that

$$\begin{gathered} \phi_n(x) \leq \phi_{n+1}(x) \text{ for all } n \in \mathbb{N} \text{ and } x \in \mathbb{R}_+, \\ \sum_{n=1}^{\infty} \phi_n(x) \text{ diverges for all } x > 0. \end{gathered} \tag{2.1}$$

If $I_n = [a_n, b_n]$ is a subinterval of the interval $I = [a, b]$ $(n = 1, 2, \ldots)$ we write $f(I_n) = f(b_n) - f(a_n)$. Any collection $\{I_n\}$ of intervals mentioned shall be nonoverlapping subintervals of the domain of the function.

Definition 2.1. Let $\Phi \subset \mathcal{F}$ and $(X, |\cdot|)$ be a real normed space. A function $f \in X^I$ is of bounded Φ-variation in the sense of Schramm in I, if

$$v_\Phi(f) = v_\Phi(f, I) := \sup \sum_n \phi_n\big(|f(I_n)|\big) < \infty, \tag{2.2}$$

where the supremum is taken over all $\{I_n\}$ where $I_n \subseteq I, n \in \mathbb{N}$ ([17, p. 51]).

The set of all functions of bounded Φ-variation in the Schramm sense will be denoted by $V_\Phi(I, \mathbb{R})$.

Various spaces of the functions of generalized bounded variation which have been considered can be obtained by making special choices of the functions ϕ_n, $n = 1, 2, \ldots$ If we take $\phi_n(x) = x$ for all n, then $V_\Phi = BV$ and the condition (2.2) coincides with the classical concept of variation in the sense of Jordan. In the case where $\Lambda = \{\lambda_n\}$ is a λ-sequence in the sense of Waterman, and $\phi_n(x) = x/\lambda_n$, we have $V_\Phi = \Lambda BV$. For $\phi_n = \varphi$, the condition (2.2) coincides with the classical concept of variation in the sense of Wiener [18].

It is known that for all $a, b, c \in I$, $a \leq c \leq b$ we have $v_\Phi(f, [a, c]) \leq v_\Phi(f, [a, b])$ (that is, v_Φ is increasing with respect to the interval) and $v_\Phi(f, [a, c]) + v_\Phi(f, [c, b]) \leq v_\Phi(f, [a, b])$.

The set of all functions of bounded Φ-variation in the Schramm sense is a symmetric and convex set; but it is not necessarily a linear space.

In what follows we denote by $\Phi BV(I, X)$ the linear space of all functions $f \in X^I$ such that $v_\Phi(\lambda f) < \infty$ for some constant $\lambda > 0$. In the space $\Phi BV(I, X)$, the function $\|\cdot\|_\Phi$ defined by

$$\|f\|_\Phi := |f(a)| + p_\Phi(f), \quad f \in \Phi BV(I, X),$$

where a is the left endpoint of I and

$$p_\Phi(f) := p_\Phi(f, I) = \inf \Big\{\epsilon > 0 : v_\Phi\big(f/\epsilon\big) \leq 1\Big\}, \quad f \in \Phi BV(I, X), \tag{2.3}$$

is a norm (see for instance [17]).

For $X = \mathbb{R}$, the linear normed space $(\Phi BV(I,\mathbb{R}), \|\cdot\|_\Phi)$ was studied by Schramm ([17, Theorem 2.3]). Hernández [5] has shown that the space $(\Phi BV(I,\mathbb{R}), \|\cdot\|_\Phi)$ is a Banach algebra. The functional $p_\Phi(\cdot)$ defined by (2.3) is called *the Luxemburg-Nakano-Orlicz seminorm* [8, 14, 16].

In the sequel, the symbol $\Phi BV(I,C)$ stands for the set of all functions $f \in \Phi BV(I,X)$ such that $f : I \longrightarrow C$ and C is a subset of X.

Lemma 2.2. *For $f \in \Phi BV(I,X)$, we have:*

(a) *if $t, t' \in I$, then $|f(t) - f(t')| \le \phi_n^{-1}(1) p_\Phi(f)$,*
 (a1) *if $f \in \Phi BV(I,X)$, then f is bounded,*
(b) *if $p_\Phi(f) > 0$ then $v_\Phi\big(f/p_\Phi(f)\big) \le 1$,*
(c) *for $\lambda > 0$,*
 (c1) *$p_\Phi(f) \le \lambda$ if and only if $v_\Phi\big(f/\lambda\big) \le 1$,*
 (c2) *if $v_\Phi\big(f/\lambda\big) = 1$ then $p_\Phi(f) = \lambda$.*

Proof. (a) Take a $\lambda > p_\Phi(f)$. Then for any $t, s \in I$ and for any finite collection $\{I_n\}$, by (2.2) and (2.3), we have

$$\phi_n\left(\frac{|f(t) - f(s)|}{\lambda}\right) \le v_\Phi\left(\frac{f}{\lambda}\right) \le 1,$$

whence, taking the inverse function ${\phi_n}^{-1}$, we obtain (a).

(a1) Since $f \in \Phi BV(I,X)$, we have $|f(t)| \le |f(a)| + |f(t) - f(a)|$ for all $t \in I$. Therefore,

$$\|f\|_\infty \le |f(a)| + \phi_n^{-1}(1) p_\Phi(f) < \infty.$$

(b) Suppose that a sequence of numbers $\lambda_n > \lambda = p_\Phi(f)$ converges to λ as $n \to \infty$. It follows from the definition of the number λ that $v_\Phi(\frac{f}{\lambda_n}) \le 1$ for all positive integers n. Since f/λ_n pointwise converges to f/λ on I as $n \to \infty$, by the lower semicontinuity of the functional $v_\Phi(\cdot)$, we obtain that

$$v_\Phi(f/\lambda) \le \lim_{n\to\infty} v_\Phi(f/\lambda_n) \le 1.$$

(c) To prove (c1), it suffices to show that if $0 < p_\Phi(f) < \lambda$, then $v_\Phi(f/\lambda) < 1$, and this is directly implied by the convexity of $v_\Phi(\cdot)$ and of part (b), that is,

$$v_\Phi(f/\lambda) \le \frac{p_\Phi(f)}{\lambda} v_\Phi\left(\frac{f}{p_\Phi(f)}\right) \le \frac{p_\Phi(f)}{\lambda} < 1.$$

To prove the second assertion (c2), it suffices to observe that neither the case $p_\Phi(f) > \lambda$ nor $p_\Phi(f) < \lambda$ occurs. □

Let $I^- := (a, b]$. If $(X, |\cdot|)$ is a Banach space and $f \in \Phi BV(I,X)$, then $f^-(t) := \lim_{s\uparrow t} f(s),\quad t \in I^-$, exists ([2, Lemma 6.10]) and is called the *left regularization* of f.

Let $\Phi BV^-(I,X)$ denote the subset in $\Phi BV(I,X)$ which consists of those functions that are left continuous on I^-.

Applying Lemma 6.12 from [2] with $(M,d,+) = (X, |\cdot|)$ we get the following lemma.

Lemma 2.3. *If X is a Banach space and $f \in \Phi BV(I,X)$, then $f^- \in \Phi BV^-(I,X)$.*

Thus, if a function has a bounded Φ-variation, then its left regularization is a left continuous function.

3. THE COMPOSITION OPERATOR

Our main result reads as follows.

Theorem 3.1. *Let $(X, |\cdot|_X)$ be a real normed space, $(Y, |\cdot|_Y)$ a real Banach space, $C \subset X$ a closed convex set and suppose that $\varphi \in \mathcal{F}$ and $h : I \times C \longrightarrow Y$. If a composition operator $H : C^I \longrightarrow Y^I$ generated by h, maps $\Phi BV(I,C)$ into $\Phi BV(I,Y)$ and is uniformly continuous, then the left regularization of h, i.e., the function $h^- : I^- \times C \longrightarrow Y$ defined by*

$$h^-(t,y) := \lim_{s \uparrow t} h(s,y), \quad t \in I^-, \quad y \in C,$$

exists and

$$h^-(t,y) = A(t)y + B(t), \quad t \in I^-, \quad y \in C,$$

for some $A : I^- \longrightarrow \mathcal{L}(X,Y)$[1], $A(\cdot)y \in \Phi BV(I^-,Y)$ and $B \in \Phi BV(I^-,Y)$. Moreover, the functions A and B are left-continuous in I^-.

Proof. For every $y \in C$, the constant function $f(t) = y$ $(t \in I)$ belongs to $\Phi BV(I,C)$. Since H maps $\Phi BV(I,C)$ into $\Phi BV(I,Y)$, it follows that the function $t \mapsto h(t,y)$ $(t \in I)$ belongs to $\Phi BV(I,Y)$. Now, by Lemma 2.3, the completeness of $(Y, |\cdot|_Y)$ implies the existence of the left regularization h^- of h.

By assumption, H is uniformly continuous on $\Phi BV(I,C)$. Let $\omega : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ be the *modulus of continuity* of H, that is,

$$\omega(\rho) := \sup\{\|H(f_1) - H(f_2)\|_\Phi : \|f_1 - f_2\|_\Phi \le \rho,\ f_1, f_2 \in \Phi BV(I,C)\} \quad \text{for} \quad \rho > 0.$$

Hence we get

$$\|H(f_1) - H(f_2)\|_\Phi \le \omega\big(\|f_1 - f_2\|_\Phi\big) \quad \text{for} \quad f_1, f_2 \in \Phi BV(I,C). \tag{3.1}$$

From the definition of the norm $\|\cdot\|_\Phi$, we obtain

$$p_\Phi\big(H(f_1) - H(f_2)\big) \le \|H(f_1) - H(f_2)\|_\Phi \quad \text{for} \quad f_1, f_2 \in \Phi BV(I,C). \tag{3.2}$$

From (3.1), (3.2) and Lemma 2.2 (c1), if $\omega\big(\|f_1 - f_2\|_\Phi\big) > 0$, then

$$v_\Phi\left(\frac{H(f_1) - H(f_2)}{\omega\big(\|f_1 - f_2\|_\Phi\big)}\right) \le 1. \tag{3.3}$$

[1] $\mathcal{L}(X,Y)$ denotes the space of all continuous linear mappings $A : X \longrightarrow Y$.

Therefore, for any $\alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \ldots < \alpha_m < \beta_m, \alpha_i, \beta_i \in I$, $i \in \{1, 2, \ldots, m\}$, $m \in \mathbb{N}$, the definitions of the operator H and the functional $v_\Phi(\cdot)$ imply

$$\sum_{i=1}^{\infty} \phi_i \left(\frac{|h(\beta_i, f_1(\beta_i)) - h(\beta_i, f_2(\beta_i)) - h(\alpha_i, f_1(\alpha_i)) + h(\alpha_i, f_2(\alpha_i))|}{\omega(\|f_1 - f_2\|_\Phi)} \right) \leq 1. \tag{3.4}$$

For $\alpha, \beta \in \mathbb{R}$, $\alpha < \beta$, we define functions $\eta_{\alpha,\beta} : \mathbb{R} \longrightarrow [0, 1]$ by

$$\eta_{\alpha,\beta}(t) := \begin{cases} 0 & \text{if} \quad t \leq \alpha, \\ \dfrac{t - \alpha}{\beta - \alpha} & \text{if} \quad \alpha \leq t \leq \beta, \\ 1 & \text{if} \quad \beta \leq t\,. \end{cases} \tag{3.5}$$

Let us fix $t \in I^-$. For arbitrary finite sequence $\inf I < \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \ldots < \alpha_m < \beta_m < t$ and $y_1, y_2 \in C$, $y_1 \neq y_2$, the functions $f_1, f_2 : I \longrightarrow X$ defined by

$$f_\ell(\tau) := \frac{1}{2}\big(\eta_{\alpha_i,\beta_i}(\tau)(y_1 - y_2) + y_\ell + y_2\big), \quad \tau \in I, \quad \ell = 1, 2, \tag{3.6}$$

belong to the space $\Phi BV(I, C)$. From (3.6) we have

$$f_1(\cdot) - f_2(\cdot) = \frac{y_1 - y_2}{2}.$$

Therefore,

$$\|f_1 - f_2\|_\Phi = \left|\frac{y_1 - y_2}{2}\right|.$$

Moreover,

$$f_1(\beta_i) = y_1, \quad f_2(\beta_i) = \frac{y_1 + y_2}{2}, \quad f_1(\alpha_i) = \frac{y_1 + y_2}{2}, \quad f_2(\alpha_i) = y_2.$$

Using (3.4), we hence get

$$\sum_{i=1}^{\infty} \phi_i \left(\frac{\left|h(\beta_i, y_1) - h\Big(\beta_i, \dfrac{y_1 + y_2}{2}\Big) - h\Big(\alpha_i, \dfrac{y_1 + y_2}{2}\Big) + h(\alpha_i, y_2)\right|}{\omega\Big(\Big|\dfrac{y_1 - y_2}{2}\Big|\Big)} \right) \leq 1. \tag{3.7}$$

Fix a positive integer m. We have

$$\sum_{i=1}^{m} \phi_i \left(\frac{|h(\beta_i, y_1) - h\left(\beta_i, \frac{y_1+y_2}{2}\right) - h\left(\alpha_i, \frac{y_1+y_2}{2}\right) + h(\alpha_i, y_2)|}{\omega\left(|\frac{y_1-y_2}{2}|\right)} \right) \leq 1. \tag{3.8}$$

From the continuity of ϕ_i, passing to the limit in (3.8) when $\alpha_1 \uparrow t$, we obtain that

$$\sum_{i=1}^{m} \phi_i(x) \leq 1 \text{ for } m = 1, 2, \ldots, \tag{3.9}$$

where

$$x = \frac{\left|h^-(t,y_1) - 2h^-\left(t, \frac{y_1+y_2}{2}\right) + h^-(t,y_2)\right|}{\omega\left(\left|\frac{y_1-y_2}{2}\right|\right)}.$$

Condition (3.9) implies that

$$\sum_{i=1}^{\infty} \phi_i(x) \le 1$$

and, by (2.1), $x = 0$, i.e.,

$$h^-\left(t, \frac{y_1+y_2}{2}\right) = \frac{h^-(t,y_1) + h^-(t,y_2)}{2}$$

for all $t \in I^-$ and all $y_1,\ y_2 \in C$.

Thus, for each $t \in I^-$, the function $h^-(t,\cdot)$ satisfies the Jensen functional equation in C. Modifying a little the standard argument (cf. Kuczma [7]), we conclude that, for each $t \in I^-$, there exist an additive function $A(t) : X \to Y$ and $B(t) \in Y$ such that $h^-(t,y) = A(t)y + B(t)$.

The uniform continuity of the operator $H : BV_\varphi(I,C) \longrightarrow BV_\varphi(I,Y)$ implies the continuity of the additive function $A(t)$. Consequently, $A(t) \in \mathcal{L}(X,Y)$ for each $t \in I^-$. Moreover, $A(\cdot)y, B(\cdot) \in \Phi BV(I^-,Y)$ for $y \in C$, since $h^-(\cdot,y) \in \Phi BV(I^-,Y)$. $\square$

Remark 3.2. Obviously, the counterpart of Theorem 3.1 for the right regularization h^+ of h defined by

$$h^+(t,y) := \lim_{s \downarrow t} h(s,y), \quad t \in I^+ := I \setminus \{\sup I\},$$

is also true.

Remark 3.3. Taking $X = Y = \mathbb{R}$, $\phi_n = \varphi := id|_{[0,+\infty)}, n \in \mathbb{N}$, in Theorem 3.1 and $C := J$, where $J \subset \mathbb{R}$ is an interval, we obtain the main result from [4].

Remark 3.4. Theorem 3.1 extends also the result of [4].

Remark 3.5. In the proof of Theorem 3.1 we apply the uniform continuity of the operator H only on the set of functions $U \subset \Phi BV_\varphi(I,C)$ such that $f \in U$ if and only if there are $\alpha, \beta \in I$, $\alpha < \beta$, such that

$$f(t) = \frac{1}{2}\Big[\eta_{\alpha,\beta}(t)(y_1 - y_2) + y + y_2\Big], \quad t \in I,$$

where $\eta_{\alpha,\beta}$ is defined by (3.4), $y_1, y_2 \in C$ and $y = y_1$ or $y = y_2$.

Thus the assumption of the uniform continuity of H on $\Phi BV(I,C)$ in Theorem 2.1 can be replaced by a weaker condition of the uniform continuity of H on U.

4. LOCALLY DEFINED OPERATORS

It is well known that every Nemytskii composition operator is locally defined (cf. [1], and also [12,19,20]). To recall the definition of a locally defined operator assume that $\mathcal{G} = \mathcal{G}(I,\mathbb{R})$ and $\mathcal{H} = \mathcal{H}(I,\mathbb{R})$ are two classes of functions $\varphi : I \to \mathbb{R}$, where $I \subset \mathbb{R}$ is an interval. A mapping $K : \mathcal{G} \to \mathcal{H}$ is said to be a locally defined operator or $(\mathcal{G},\mathcal{H})$-local operator if for any open interval $J \subset \mathbb{R}$ and for any functions $\varphi, \psi \in \mathcal{G}$,

$$\varphi\big|_{J\cap I} = \psi\big|_{J\cap I} \Rightarrow K(\varphi)\big|_{J\cap I} = K(\psi)\big|_{J\cap I},$$

where $\varphi|_{J\cap I}$ denotes the restriction of φ to $J \cap I$.

The form of the locally defined operator strongly depends on the nature of the function spaces $\mathcal{G}$ and $\mathcal{H}$ which are its domains and ranges, respectively.

Let $C(I)$ be a family of real continuous functions defined on I and $CM_+(I)$ and $CM_-(I)$ denote, respectively, a family of continous nondecreasing and continuous nonincreasing functions $f : I \to \mathbb{R}$.

We write $CBV(I)$ for $C(I) \cap BV(I,\mathbb{R})$.

Proposition 4.1. *If a locally defined operator K maps $CBV(I)$ into $CM_+(I)$, then it is constant, that is, there is a function $b \in CM_+(I)$ such that*

$$K(\varphi) = b, \qquad \varphi \in CBV(I).$$

Proof. Let $K : CBV(I) \to CM_+(I)$ be a locally defined operator. Since $CM_+(I) \subset CBV(I)$ and $CM_-(I) \subset CBV(I)$, an operator K is $(CM_+, CM_+)-$ and $(CM_-, CM_+)-$ local. Hence, K is the Nemytskii composition operator and by Theorem 1 and Theorem 4 from [20], we get our claim. □

Similarly, by [16, Remark 4], we can get the following proposition.

Proposition 4.2. *If a locally defined operator K maps $CBV(I)$ into $CM_-(I)$, then it is constant, that is, there is a function $b \in CM_-(I)$ such that*

$$K(\varphi) = b, \qquad \varphi \in CBV(I).$$

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