ON A NONLINEAR INTEGRODIFFERENTIAL EVOLUTION INCLUSION WITH NONLOCAL INITIAL CONDITIONS IN BANACH SPACES

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Abstract. In this paper, we discuss the existence results for a class of nonlinear integrodifferential evolution inclusions with nonlocal initial conditions in Banach spaces. Our results are based on a fixed point theorem for condensing maps due to Martelli and the resolvent operators combined with approximation techniques.

Keywords: nonlinear integrodifferential evolution inclusions, fixed point, resolvent operator, nonlocal initial condition.

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1. INTRODUCTION

This paper is concerned mainly with the existence of mild solutions for first-order nonlinear evolution integrodifferential inclusions with nonlocal initial condition

$$\begin{aligned} &x'(t) \in A(t)\left[x(t) + \int_0^t H(t,s)x(s)ds\right] + \\ &\qquad + F\left(t, x(\sigma_1(t)), \ldots, x(\sigma_n(t)), \int_0^t h(t,s,x(\sigma_{n+1}(s)))ds\right), \quad t \in J, \\ &x(0) + g(x) = x_0, \end{aligned} \tag{1.1}$$

where $J = [0,b]$, the state $x(\cdot)$ takes values in a Banach space X with the norm $|\cdot|$ and $A(t)$ is a closed linear operator on X with dense domain $D(A)$, which is independent of t. $H(t,s), t,s \in J$, is a bounded operator in X. $F : J \times X^{n+1} \to \mathcal{P}(X)$ is a multivalued map, $\mathcal{P}(X)$ is the family of all subsets of X. $h : \Delta \times X \to X$,

$\Delta = \{(t, s) : 0 \leq s \leq t \leq b\}$, $g : C(J, X) \to X$, $\sigma_i : J \to J, i = 1, \ldots, n+1$, are given functions to be specified later.

The nonlocal Cauchy problem was studied by Byszewski [8,9], and subsequently, as it can be applied in physics with better than the classical initial condition, it has been studied extensively under various conditions on $A(A(t)), F$ and g; see [1, 10, 12, 25, 26] and references therein. Recently, the existence of solutions for some classes of abstract integrodifferential equations and integrodifferential inclusions with nonlocal conditions have been investigated by many authors. For example, Balachandran *et al.* [2] have studied the nonlinear time varying delay integrodifferential equations of Sobolev type with nonlocal conditions. Liang and Xiao [21] have established some new theorems about the existence and uniqueness of solutions for the semilinear integrodifferential equations with nonlocal initial conditions. Lin and Liu [22] have discussed the nonlocal Cauchy problem for semilinear integrodifferential equations by using resolvent operators. Liu [23] have obtained the representation of weak solutions of Cauchy problem for integrodifferential evolution equations in abstract spaces. Kumar [18] has proved the existence of solutions for nonlocal neutral integrodifferential equations in Banach spaces by using the theory of analytic resolvent operators. Yan [29, 30] have established a sufficient condition for the existence of mild solutions of nonlinear functional integrodifferential equations with nonlocal conditions in Banach spaces. Benchohra and Ntouyas [5] have studied nonlocal Cauchy problems for neutral functional differential and integrodifferential inclusions in Banach spaces.

However, most of the previous research on nonlocal Cauchy problems was based on the contraction mapping principle. This condition turns out to be quite restrictive. The purpose of this paper is to prove the existence of mild solutions for nonlinear integrodifferential inclusions (1.1) by relying on a fixed-point theorem for condensing maps due to Martelli [24]. Our main condition is only concerned with the continuous and Carathéodory conditions. Indeed, we only require that F satisfies the Carathéodory condition. Moreover, we also have consider the case in which g is continuous but without imposing severe compactness conditions and convexity.

The rest of this paper is organized as follows. In Section 2, we will recall briefly some preliminary facts which will be used in paper. Section 3 is devoted to the existence of mild solutions of the problem (1.1). In Section 4, we present an example illustrating the abstract theory of the previous sections. Finally in Section 5, we apply the preceding technique to a controlled problem.

2. PRELIMINARIES

In this section, we introduce some basic definitions, notations and lemmas which are used throughout this paper.

Let $C(J, X)$ denote the Banach space of continuous functions from J into X with the norm

$$\|x\|_\infty = \sup\{|x(t)| : t \in J\}$$

and let $B(X)$ denote the Banach space of bounded linear operators from X into itself.

A measurable function $x : J \to X$ is Bochner integrable if and only if $|x|$ is Lebesgue integrable (for properties of the Bochner integral see Yosida [31]). $L^1(J, X)$ denotes the linear space of equivalence classes of all measurable functions $x : J \to X$, which are normed by

$$\|x\|_{L^1} = \int_0^b |x(t)|dt \quad \text{for all} \quad x \in L^1(J, X).$$

Let $(X, |\cdot|)$ be a Banach space. A multivalued map $G : X \to \mathcal{P}(X)$ is convex (closed) valued if $G(X)$ is convex (closed) for all $x \in X$. G is bounded on bounded sets if $G(B) = \bigcup_{x\in B} G(x)$ is bounded in X for any bounded set B of X, that is, $\sup_{x\in B}\{\sup\{|y| : y \in G(x)\}\} < \infty$.

G is called upper semicontinuous (u.s.c.) on X if, for each $x_0 \in X$, the set $G(x_0)$ is a nonempty, closed subset of X and if, for each open set N of X containing $G(x_0)$, there exists an open neighborhood V of x_0 such that $G(V) \subseteq N$.

The multivalued operator G is called compact if $\overline{G(X)}$ is a compact subset of X. G is said to be completely continuous if $G(D)$ is relatively compact for every bounded subset D of X. If the multi-valued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph, i.e., $x_n \to x_*, y_n \to y_*, y_n \in G(x_n)$ imply $y_* \in G(x_*)$.

In what follows $BCC(X)$ denotes the set of all nonempty bounded, closed and convex subsets of X.

A multivalued map $G : J \to BCC(X)$ is said to be measurable if, for each $x \in X$, the function $k : J \to \mathbb{R}$, defined by

$$k(t) = d(x, G(t)) = \inf\{|x - z| : z \in G(t)\},$$

belongs to $L^1(J, \mathbb{R})$.

An upper semicontinuous map $G : X \to \mathcal{P}(X)$ is said to be condensing if, for any subset $B \subseteq X$ with $\alpha(B) \neq 0$, we have $\alpha(G(B)) < \alpha(B)$, where α denotes the Kuratowski measure of noncompactness [4].

G has a fixed point if there is $x \in X$ such that $x \in G(x)$. For more details on multivalued maps see the books of Deimling [11], and Hu and Papageorgiou [14].

Definition 2.1. A resolvent operator for problem (1.1) is a bounded operator valued function $R(t, s) \in B(X), 0 \le s \le t \le b$, the space of bounded linear operators on X, having the following properties:

(a) $R(t, s)$ is strongly continuous in s and t, $R(s, s) = I, 0 \le s \le b, \|R(t, s)\| \le Me^{\beta(t-s)}$ for some constants M and β.

(b) $R(t, s)Y \subset Y, R(t, s)$ is strongly continuous in s and t on Y, and Y is the Banach space formed from $D(A)$, the domain of $A(t)$, endowed with the graph norm.

(c) For each $x \in X, R(t, s)x$ is continuously differentiable in $t, s \in J$ and

$$\frac{\partial R}{\partial t}(t, s)x = A(t)\left[R(t, s)x + \int_s^t H(t, \tau)R(\tau, s)xd\tau\right].$$

The main tool in our approach is the following fixed-point theorem due to Martelli.

Lemma 2.2 (Martelli [24]). *Let X be a Banach space and let $G : X \to BCC(X)$ be a condensing map. If the set*

$$\Omega = \{x \in X : \lambda x \in \lambda Gx \text{ for some } 0 < \lambda < 1\}$$

is bounded, then G has a fixed point.

Remark 2.3. We remark that a completely continuous multivalued map is the easiest example of a condensing map.

3. EXISTENCE OF MILD SOLUTIONS

In this section we give our main existence result for the problem (1.1). Now, we can define the mild solution of the problem (1.1).

Definition 3.1. A continuous function $x(t)$ satisfying the following integral inclusion:

$$\begin{aligned} x(t) \in R(t,0)[x_0 - g(x)] + \int_0^t R(t,s)\times \\ \times F\left(s, x(\sigma_1(s)), \dots, x(\sigma_n(s)), \int_0^s h(s,\tau,x(\sigma_{n+1}(\tau)))d\tau\right)ds \end{aligned} \tag{3.1}$$

is called a mild solution of the problem (1.1) on J.

Further we assume the following hypotheses:

(H1) The resolvent operator $R(t,s)$ is compact for $t, s > 0$.
(H2) For each $(t,s) \in \Delta$, the function $h(t,s,\cdot) : X \to X$ is continuous and for each $x \in X$ the functions $h(\cdot,\cdot,x) : \Delta \to X$ is strongly measurable.
(H3) $F : J \times X^{n+1} \to BCC(X)$ is measurable to t for each $(x_1, \dots, x_{n+1}) \in X^{n+1}$, u.s.c. with respect to $(x_1, \dots, x_{n+1}) \in X^{n+1}$ for each $t \in J$, and $x \in C(J,X)$ the set

$$\begin{aligned} S_{F,x} = \Bigg\{ & f \in L^1(J,X) : \\ & f(t) \in F\left(t, x(\sigma_1(t)), \dots, x(\sigma_n(t)), \int_0^t h(t,s,x(\sigma_{n+1}(s)))ds\right)\Bigg\} \end{aligned}$$

is nonempty.
(H4) There exists a positive function $p : J \to [0,\infty)$ and, for every $s \in [0,t]$, the function $s \mapsto e^{-\beta s}p(s)$ belongs to $L^1([0,t],\mathbb{R}^+)$ such that

$$\begin{aligned} \|F(t,x_1,\dots,x_{n+1})\| := \{|f| : f(t) \in F(t,x_1,\dots,x_{n+1}))\} \le \\ \le p(t)\Theta(\|x_1\| + \ldots + \|x_{n+1}\|), \end{aligned}$$

for a.e. $t \in J$ and each $x_i \in X, i = 1, \dots, n+1$, where $\Theta : [0, \infty) \to (0, \infty)$ is a continuous nondecreasing function.

(H5) There exists an integrable function $p_0 : J \times J \to [0, \infty)$ such that

$$\|h(t, s, x)\| \le p_0(t, s)\Theta_0(\|x\|), \quad t, s \in J, x \in X,$$

where $\Theta_0 : [0, \infty) \to (0, \infty)$ is a continuous nondecreasing function.

(H6) $\sigma_i : J \to J, i = 1, \dots, n+1$, are continuous functions.

(H7) The function $g(\cdot) : C(J, X) \to X$ is continuous and there exists a $\delta \in (0, b)$ such that $g(\phi) = g(\psi)$ for any $\phi, \psi \in C := C(J, X)$ with $\phi = \psi$ on $[\delta, b]$.

(H8) (i) There is a constant $c > 0$ such that

$$0 \le \limsup_{\|\phi\|_\infty \to \infty} \frac{|g(\phi)|}{\|\phi\|_\infty} \le c, \qquad \phi \in C. \tag{3.2}$$

(ii) The following inequality holds:

$$M_0^2 c < 1, \tag{3.3}$$

where $M_0 = M \max\{1, e^{\beta b}\}$.

Lemma 3.2 ([20])**.** *Let J be a compact real interval and let X be a Banach space. Let F be a multivalued map satisfying* (H3) *and let Γ be a linear continuous operator from $L^1(J, X)$ to $C(J, X)$. Then the operator*

$$\Gamma \circ S_F : C(J, X) \to BCC(C(J, X)), \quad x \to (\Gamma \circ S_F)(x) := \Gamma(S_F, x)$$

is a closed graph in $C(J, X) \times C(J, X)$.

Theorem 3.3. *If hypotheses* (H1)–(H8) *are satisfied, then the nonlocal Cauchy problem* (1.1) *has at least one mild solution on J provided that*

$$\int_1^\infty \frac{1}{s + \Theta(s) + \Theta_0(s)} ds = \infty. \tag{3.4}$$

Proof. We transform the problem (1.1) into a fixed point problem. Consider the multi-valued map $P : C(J, X) \to \mathcal{P}(X)$ defined by

$$P(x) := \left\{ \rho \in C(J, X) : \rho(t) = R(t, 0)[x_0 - g(x)] + \int_0^t R(t - s) f(s) ds, f \in S_{F,x} \right\}$$

has a fixed point. This fixed point is then a mild solution of the problem (1.1).

Let $\{\delta_n : n \in \mathbb{N}\}$ be a decreasing sequence in $(0, b)$ such that $\lim_{n\to\infty} \delta_n = 0$. To prove the above problem, we consider the following inclusion:

$$\begin{aligned} &x'(t) \in A(t)\left[x(t) + \int_0^t H(t,s)x(s)ds\right] + \\ &\qquad + F\left(t, x(\sigma_1(t)), \ldots, x(\sigma_n(t)), \int_0^t h(t,s,x(\sigma_{n+1}(s)))ds\right), \quad t \in J, \\ &x(0) + R(\delta_n, 0)g(x) = x_0, \end{aligned} \tag{3.5}$$

has at least one mild solution $x_n \in C(J, X)$.

For fixed $n \in \mathbb{N}$, set $P_n : C(J, X) \to \mathcal{P}(X)$ defined by

$$\begin{aligned} P_n(x) := \Big\{ &\rho_n \in C(J,X) : \rho_n(t) = \\ &= R(t,0)[x_0 - R(\delta_n,0)g(x)] + \int_0^t R(t,s)f(s)ds, f \in S_{F,x} \Big\} \end{aligned}$$

for $t \in [0, b]$. It is easy to see that the fixed point of P_n is the mild solution of the nonlocal Cauchy problem (3.5). We now show that P_n satisfies all conditions of Lemma 2.2. The proof will be given in several steps.

Step 1. $P_n(x)$ is convex for each $x \in C(J, X)$.

In fact, if $\rho_n^1, \rho_n^2 \in P_n(x)$, then there exist $f_1, f_2 \in S_{F,x}$ such that for each $t \in J$ we have

$$\rho_n^i(t) = R(t,0)[x_0 - R(\delta_n,0)g(x)] + \int_0^t R(t,s)f_i(s)ds, \ i = 1, 2. \tag{3.6}$$

Let $0 \le \lambda \le 1$, then, for each $t \in J$ we have

$$\begin{aligned} (\lambda\rho_n^1 + (1-\lambda)\rho_n^2)(t) &= R(t,0)[x_0 - R(\delta_n,0)g(x)] + \\ &+ \int_0^t R(t,s)(\lambda f_1(s) + (1-\lambda)f_2(s))ds. \end{aligned}$$

Since $S_{F,x}$ is convex (because F has convex values) we have

$$\lambda\rho_n^1 + (1-\lambda)\rho_n^2 \in P_n(x).$$

Step 2. $P_n(x)$ maps bounded sets into bounded sets in $C(J, X)$.

In fact, we need only to show that there exists a positive constant d such that, for each $\rho_n \in P_n(x)$, $x \in B_q := \{x \in C([0, b], X) : \|x\|_\infty \le q\}$, we obtain $\|\rho_n\|_\infty \le d$. If $\rho_n \in P_n(x)$, then there exists $f \in S_{F,x}$ such that, for each $t \in J$, we have

$$\rho_n(t) = R(t,0)[x_0 - R(\delta_n,0)g(x)] + \int_0^t R(t,s)f(s)ds. \tag{3.7}$$

However, on the other hand, from condition (H8), we conclude that there exist positive constants ϵ and γ such that, for all $\|\phi\|_\infty > \gamma$,

$$|g(\phi)| \le (c+\epsilon)\|\phi\|_\infty, \quad M_0^2(c+\epsilon) < 1. \tag{3.8}$$

Let

$$\begin{aligned} &E_1 = \{\phi : \|\phi\|_\infty \le \gamma\}, \quad E_2 = \{\phi : \|\phi\|_\infty > \gamma\},\\ &C_1 = \max\{|g(\phi)|, \phi \in E_1\}. \end{aligned}$$

Therefore,

$$|g(\phi)| \le C_1 + (c+\epsilon)\|\phi\|_\infty. \tag{3.9}$$

It is from (H2), (H4)–(H6), (3.8) and (3.9) that for each $t \in [0,b]$ we have

$$\begin{aligned}
&|\rho_n(t)| \le \\
&\le |R(t,0)[x_0 - R(\delta_n,0)g(x)]| + \left|\int_0^t R(t,s)f(s)ds\right| \le \\
&\le Me^{\beta t}[|x_0| + M|g(x)|] + M_0\int_0^t e^{\beta(t-s)}|f(s)|ds \le \\
&\le M_0[|x_0| + M_0(C_1 + (c+\epsilon)\|x\|_\infty)] + \\
&\quad + M_0\int_0^t e^{-\beta s}p(s)\Theta\left[|x(\sigma_1(s))| + \ldots + |x(\sigma_n(s))| + \int_0^s |h(s,\tau,x(\sigma_{n+1}(\tau)))|d\tau\right]ds \le \\
&\le M_0[|x_0| + M_0(C_1 + (c+\epsilon)q)] + \\
&\quad + M_0\int_0^t e^{-\beta s}p(s)\Theta\Bigg[\sup_{\tau\in[0,b]}|x(s)| + \ldots + \sup_{s\in[0,b]}|x(s)| + \\
&\qquad\qquad + \int_0^s p_0(s,\tau)\Theta_0(|x(\sigma_{n+1}(\tau))|)d\tau\Bigg]ds \le \\
&\le M_0[|x_0| + M_0(C_1 + (c+\epsilon)q)] + \\
&\quad + M_0\int_0^t e^{-\beta s}p(s)\Theta\left[n\sup_{s\in[0,b]}|x(s)| + \int_0^s p_0(s,\tau)\Theta_0(\sup_{\tau\in[0,b]}|x(\tau)|)d\tau\right]ds \le \\
&\le M_0[|x_0| + M_0(C_1 + (c+\epsilon)q)] + \\
&\quad + M_0\Theta\left[nq + \Theta_0(q)\int_0^b p_0(s,s)ds\right]\int_0^b e^{-\beta s}p(s)ds =: d.
\end{aligned}$$

Thus, for each $\rho_n \in P_n(B_q)$, $\|\rho_n\|_\infty \le d$.

Step 3. P_n sends bounded sets into equicontinuous sets of $C(J,X)$.

Let $0 < t_1 < t_2 \le b$, B_q be a bounded set as in Step 2. For each $x \in B_q$ and $\rho_n \in P_n(x)$, then there exists $f \in S_{F,x}$ such that for each $t \in J$, we have

$$\rho_n(t) = R(t,0)[x_0 - R(\delta_n,0)g(x)] + \int_0^t R(t,s)f(s)ds. \tag{3.10}$$

In the view of (3.10) and (H1)–(H5), we have

$$\begin{aligned}
&|\rho_n(t_2) - \rho_n(t_1)| \le \\
&\quad \le |[R(t_2,0) - R(t_1,0)][x_0 - R(\delta_n,0)g(x)]| + \\
&\quad + \int_0^{t_1} |R(t_2,s) - R(t_1,s)f(s)|ds \ + \int_{t_1}^{t_2} |R(t_2,s)f(s)|ds \le \\
&\quad \le |R(t_2,0) - R(t_1,0)||[x_0 - R(\delta_n,0)g(x)]| + \\
&\quad + \frac{M_0}{M}\Theta\left[nq + \Theta_0(q)\int_0^b p_0(s,s)ds\right]\int_0^{t_1} |R(t_2,s) - R(t_1,s)|e^{-\beta s}p(s)ds + \\
&\quad + Me^{\beta t_2}\Theta\left[nq + \Theta_0(q)\int_0^b p_0(s,s)ds\right]\int_{t_1}^{t_2} e^{-\beta s}p(s)ds.
\end{aligned}$$

The right-hand side of the above inequality tends to zero independently of $x \in B_q$ as $(t_2 - t_1) \to 0$, since the compactness of $R(t,s)$ for $t, s > 0$, implies the continuity in the uniform operator topology. Thus P_n sends B_q into an equicontinuous family of functions.

As a consequence of Step 2, Step 3, together with the Ascoli-Arzela theorem, we conclude that $P_n : C(J,X) \to \mathcal{P}(X)$ is completely continuous and therefore is a condensing map.

Step 4. P_n has a closed graph.

Let $x^{(m)} \to x^*, (m \to \infty), \rho_n^{(m)} \in P_n(x^{(m)}), x^{(m)} \in B_q$ and $\rho_n^{(m)} \to \rho_n^*$. We shall show that $\rho_n^* \in P_n(x^*)$. Now $\rho_n^{(m)} \in P_n(x^{(m)})$ means that there exists $f^{(m)} \in S_{F,x^{(m)}}$ such that, for each $t \in J$,

$$\rho_n^{(m)}(t) = R(t,0)[x_0 - R(\delta_n,0)g(x^{(m)})] + \int_0^t R(t,s)f^{(m)}(s)ds.$$

We must prove that there exists $f^* \in S_{F,x^*}$ such that, for each $t \in J$,

$$\rho_n^*(t) = R(t,0)[x_0 - R(\delta_n,0)g(x^*)] + \int_0^t R(t,s)f^*(s)ds.$$

Clearly, we have that

$$\begin{aligned}&\|(\rho_n^{(m)}(t)-R(t,0)[x_0-R(\delta_n,0)g(x^{(m)})])-\\&-(\rho_n^*(t)-R(t,0)[x_0-R(\delta_n,0)g(x^*)])\|_\infty\to 0\quad\text{as}\quad m\to\infty.\end{aligned}$$

Consider the linear continuous operator $\Phi : L^1(J,X)\to C(J,X)$,

$$f\mapsto(\Phi f)(t)=\int_0^t R(t,s)f(s)ds.$$

We can see that the operator Φ is linear and continuous. Indeed, one has

$$\|\Phi f\|_\infty\le M_0\Theta\left[nq+\Theta_0(q)\int_0^b p_0(s,s)ds\right]\int_0^t e^{-\beta s}p(s)ds.$$

We can see that the operator Φ is linear and continuous. From (H3) and Lemma 3.2, it follows that $\Phi\circ S_F$ is a closed graph operator. Also, from the definition of Φ, we have that

$$\rho_n^{(m)}-R(t,0)[x_0-R(\delta_n,0)g(x^{(m)})]\in\Phi(S_{F,x^{(m)}}).$$

Since $x^{(m)}\to x^*$, for some $f^*\in S_{F,x^*}$, it follows that

$$\rho_n^*(t)-R(t,0)[x_0-R(\delta_n,0)g(x^*)]=\int_0^t R(t,s)f^*(s)ds$$

for some $f^*\in S_{F,x^*}$.
Step 5. The set $\Omega=\{x\in C(J,X):\lambda\in(0,1),x=\lambda P_n(x)\}$ is bounded.

Indeed, let $\lambda\in(0,1)$ and let $x\in C(J,X)$ be a possible solution of $x=\lambda P_n(x)$ for some $0<\lambda<1$. Then for any $x\in\Omega$, we have

$$x(t)=\lambda R(t,0)[x_0-R(\delta_n,0)g(x)]+\lambda\int_0^t R(t,s)f(s)ds \tag{3.11}$$

for some $f \in S_{F,x}$. It follows from (H1)–(H6) and (3.11) that for each $t \in [0,b]$ we have

$$e^{-\beta t}|x(t)| \leq M[|x_0| + M_0|g(x)|] + M\int_0^t e^{-\beta s}p(s)\Theta\Big[|x(\sigma_1(s))| + \ldots + |x(\sigma_n(s))| + \\ + \int_0^s |h(s,\tau,x(\sigma_{n+1}(\tau)))|d\tau\Big]ds \leq$$

$$\leq M[|x_0| + M_0(C_1 + (c+\epsilon)\|x\|_\infty)] + M\int_0^t e^{-\beta s}p(s)\Theta\Big[\sup_{s\in[0,t]}|x(s)| + \\ + \ldots + \sup_{s\in[0,t]}|x(s)| + \int_0^s p_0(s,\tau)\Theta_0(|x(\sigma_{n+1}(\tau))|)d\tau\Big]ds \leq$$

$$\leq M[\|x_0\| + M_0(C_1 + (c+\epsilon)\|x\|_\infty)] + \\ + M\int_0^t e^{-\beta s}p(s)\Theta\Big[n\sup_{s\in[0,t]}|x(s)| + \int_0^s p_0(s,\tau)\Theta_0(\sup_{\tau\in[0,s]}|x(\tau)|)d\tau\Big]ds.$$

We consider the function η defined by

$$\eta(t) := \sup\{|x(s)| : 0 \leq s \leq t\}, \quad t \in [0,b].$$

By the previous inequality and (3.8), we have for $t \in [0,b]$

$$(e^{-\beta t} - MM_0(c+\epsilon))\eta(t) \leq M[|x_0| + M_0C_1] + \\ + M\int_0^t e^{-\beta s}p(s)\Theta\Big[n\eta(s) + \int_0^s p_0(s,\tau)\Theta_0(\eta(\tau))d\tau\Big]ds.$$

Denote the right-hand side of the above inequality by $w(t)$. Then we have

$$(e^{-\beta t} - MM_0(c+\epsilon))\eta(t) \leq w(t) \quad \text{for all} \quad t \in [\delta, b]$$

and

$$w(0) = M[|x_0| + M_0C_1],$$

$$w'(t) \leq Me^{-\beta t}p(t)\Theta\Big[n\eta(t) + \int_0^t p_0(s,s)\Theta_0(\eta(s))ds\Big] \leq$$

$$\leq Me^{-\beta t}p(t)\Theta\Big[\frac{ne^{\beta t}}{1 - MM_0(c+\epsilon)e^{\beta t}}w(t) + \\ + \int_0^t p_0(s,s)\Theta_0\Big(\frac{e^{\beta s}}{1 - MM_0(c+\epsilon)e^{\beta s}}w(s)\Big)ds\Big], \quad t \in [0,b].$$

If $\xi(t) = \frac{e^{\beta t}}{1-MM_0(c+\epsilon)e^{\beta t}} w(t)$, then $\xi(0) = \frac{1}{1-MM_0(c+\epsilon)} w(0)$, $w(t) \le \xi(t)$, and

$$w'(t) \le Me^{-\beta t} p(t)\Theta\Big[n\xi(t) + \int_0^t p_0(s,s)\Theta_0(\xi(s))ds\Big].$$

Let $v(t) = n\xi(t) + \int_0^t p_0(s,s)\Theta_0(\xi(s))ds$. Then $v(0) = n\xi(0)$, $\xi(t) \le v(t)$, and we have

$$\begin{aligned} v'(t) &= n\xi'(t) + p_0(t,t)\Theta_0(\xi(t)) \le \\ &\le \frac{n\beta e^{\beta t}}{(1-MM_0(c+\epsilon)e^{\beta t})^2} w(t) + \frac{ne^{\beta t}}{1-MM_0(c+\epsilon)e^{\beta t}} w'(t) + p_0(t,t)\Theta_0(v(t)) \le \\ &\le \frac{n\beta e^{\beta t}}{(1-MM_0(c+\epsilon)e^{\beta t})^2} v(t) + \\ &\quad + \frac{Mne^{\beta t}}{1-MM_0(c+\epsilon)e^{\beta t}} e^{-\beta t} p(t)\Theta(v(t)) + p_0(t,t)\Theta_0(v(t)) \le \\ &\le m_*(t)[v(t) + \Theta(v(t)) + \Theta_0(v(t))], \end{aligned}$$

where $m_*(t) = \max\{\frac{n\beta e^{\beta t}}{(1-MM_0(c+\epsilon)e^{\beta t})^2}, \frac{Mne^{\beta t}}{1-MM_0(c+\epsilon)e^{\beta t}} e^{-\beta t} p(t), p_0(t,t)\}$. This implies for each $t \in [0,b]$ that

$$\int_{v(0)}^{v(t)} \frac{ds}{s + \Theta(s) + \Theta_0(s)} \le \int_0^t m_*(s)ds < \infty.$$

Thus from (3.4) there exists a constant d_* such that $v(t) \le d_*, t \in [0,b]$, and hence $|x| \le d^*$, where d^* depends only on the functions p, p_0, Θ, and Θ_0. This shows that Ω is bounded.

As a consequence of Lemma 2.2, we deduce that P_n has at least fixed point x_n in $C(J,X)$, which is in turn a mild solution of (3.5). Then we have

$$x_n(t) = R(t,0)[x_0 - R(\delta_n,0)g(x_n)] + \int_0^t R(t,s)f_n(s)ds, \tag{3.12}$$

for $t \in [0,b]$, and some $f_n \in S_{F,x_n}$.

Next we will show that the set $\{x_n : n \in \mathbb{N}\}$ is relatively compact in $C(J,X)$.

Step 6. $\{x_n : n \in \mathbb{N}\}$ is equicontinuous on J.

For $\varepsilon > 0, x_n \in B_q$, there exists a constant $\eta > 0$ such that for all $t \in (0,b]$ and $\xi \in (0,\eta)$ with $t + \xi \le b$ we have

$$\begin{aligned} &|x_n(t+\xi) - x_n(t)| \le \\ &\quad \le |[R(t+\xi,0) - R(t,0)][x_0 - R(\delta_n,0)g(x_n)]| + \\ &\quad\quad + M_0\Theta\Big[nq + \Theta_0(q)\int_0^b p_0(s,s)ds\Big]\int_t^{t+\xi} e^{-\beta s}p(s)ds + \\ &\quad\quad + \frac{M_0}{M}\Theta\Big[nq + \Theta_0(q)\int_0^b p_0(s,s)ds\Big]\int_0^t |R(t+\xi,s) - R(t,s)|e^{-\beta s}p(s)ds. \end{aligned}$$

Using the compactness of $R(t,s)$ for $t,s>0$, we get that

$$|[R(t+\xi,0)-R(t,0)][x_0-R(\delta_n,0)g(x_n)]|<\frac{\varepsilon}{3}, \tag{3.13}$$

and

$$\int\limits_t^{t+\xi} e^{-\beta s}p(s)ds<\frac{1}{3M_0\Theta[nq+\Theta_0(q)\int_0^b p_0(s,s)ds]}\varepsilon, \tag{3.14}$$

$$\int\limits_0^t |R(t+\xi,s)-R(t,s)|e^{-\beta s}p(s)ds<\frac{M}{3M_0\Theta[nq+\Theta_0(q)\int_0^b p_0(s,s)ds]}\varepsilon. \tag{3.15}$$

Thus by (3.13)–(3.15) one has

$$|x_n(t+\xi)-x_n(t)|<\varepsilon.$$

Therefore, $\{x_n(t):n\in\mathbb{N}\}$ is equicontinuous for $t\in(0,b]$. Clearly, $\{x_n(0):n\in\mathbb{N}\}$ is equicontinuous.

Step 7. $\{x_n(t):n\in\mathbb{N}\}$ is relatively compact in X.

Let $W(t)=\{x_n(t):x_n\in P_n(B_q)\}$. We note that $W(0)$ is relatively compact in X. Let $0<t\leq s\leq b$ be fixed and ε a real number satisfying $0<\varepsilon<t$, for $x\in B_q$, we define

$$x_n^\varepsilon(t)=R(t,0)[x_0-R(\delta_n,0)g(x)]+\int\limits_0^{t-\varepsilon} R(t,s)f(s)ds$$

for some $f\in S_{F,x}$. Using the compactness of $R(t,s)$ for $t,s>0$, we obtain that the set $W_\varepsilon(t)=\{x_n^\varepsilon(t):x_n^\varepsilon\in P_n(B_q)\}$ is pre-compact in X for every $\varepsilon, 0<\varepsilon<t$. Moreover, for every $x\in B_q$ we have

$$|x_n(t)-x_n^\varepsilon(t)|\leq\int\limits_{t-\varepsilon}^{t}|R(t,s)f(s)|ds\leq$$

$$\leq M_0\Theta\left[nq+\Theta_0(q)\int\limits_0^b p_0(s,s)ds\right]\int\limits_{t-\varepsilon}^{t} e^{-\beta s}p(s)ds.$$

Therefore, there are relatively compact sets arbitrarily close to the set $W(t)=\{x_n(t):x_n\in P_n(B_q)\}$, and $W(t)$ is a relatively compact in X.

Set

$$\tilde{x}_n(t):=\begin{cases} x_n(t), & \text{if } t\in[\delta_n,b],\\ x_n(\delta_n), & \text{if } t\in[0,\delta_n].\end{cases}$$

Using condition (H7), we obtain

$$g(x_n)=g(\tilde{x}_n),$$

where $\tilde{x}_n(t) = x_n(t)$ for $t \in [\delta_n, b]$. On the other hand, in Steps 6 and 7, applying the Arzela-Ascoli's theorem again one obtains the relatively compact of $\{\tilde{x}_n : n \in \mathbb{N}\}$ in $C((0,b], X)$. Therefore there exists a subsequence of $\{\tilde{x}_n : n \in \mathbb{N}\}$ denoted again by $\{\tilde{x}_n : n \in \mathbb{N}\}$ and a function $x \in C((0,b], X)$ such that

$$\tilde{x}_n \to x \quad \text{as} \quad n \to \infty.$$

Therefore, by the continuity of $R(t,s)$ and g, we get

$$x_n(0) = x_0 - R(\delta_n, 0)g(x_n) = x_0 - R(\delta_n, 0)g(\tilde{x}_n) \to x_0 - g(x) = x(0) \ \text{ as } \ n \to \infty.$$

Thus the sequence $\{x_n(0) : n \in \mathbb{N}\}$ is relatively compact.

These facts imply that $\{x_n : n \in \mathbb{N}\}$ in $C(J, X)$ is relatively compact. Therefore, without loss of generality, we may suppose that

$$x_n \to x_* \in C(J, X) \ \text{ as } \ n \to \infty.$$

Obviously, $x_* \in C(J, X)$, taking the limit in (3.12) of both sides, we obtain

$$x_*(t) = R(t,0)[x_0 - g(x_*)] + \int_0^t R(t,s) f_*(s) ds, \tag{3.16}$$

for $t \in J$, and some $f_* \in S_{F,x_*}$, which implies that x_* is the mild solution of the problem (1.1) and the proof of Theorem 3.3 is complete. $\square$

4. EXAMPLE

Consider the following first-order partial functional integrodifferential inclusions of the form:

$$\begin{aligned} x'(t) \in {}& \frac{\partial^2}{\partial x^2}\left[a_0(t,x)z(t,x) + \int_0^t l(t,s)z(s,x)ds\right] + \\ & + \frac{z^2(\sin t, x)}{(1+t)(1+t^2)} + \int_0^t \frac{e^{z(\sin s, x)}}{(1+t)(1+t^2)^2(1+s)^2} ds, \\ z(t,0) = {}& z(t,\pi) = 0, \\ z(0,x) + {}& \sum_{i=1}^{p} c_i \sqrt[3]{z(t_i, x)} = z_0(x), \quad 0 \le t \le 1, 0 \le x \le \pi, \end{aligned} \tag{4.1}$$

where $a_0(t,x)$ are continuous and satisfy certain smoothness conditions, $0 < t_1 < t_2 < \ldots < t_p \le 1$ and c_i are constants, $z_0(x) \in X = L^2([0,\pi])$ and $z_0(0) = z_0(\pi) = 0$.

Let $X = L^2([0,\pi])$ and the operators $A(t)$ be defined by

$$A(t)w = a_0(t,x)w''$$

with the domain $D(A) = \{w \in X : w, w^{''}$ are absolutely continuous, $w^{''} \in X, w(0) = w(\pi) = 0\}$, then $A(t)$ generates an evolution system and $R(t,s)$ can be deduced from the evolution systems [15,16,27] such that $R(t,s)$ is compact and $\|R(t,s)\| \le Me^{\beta(t-s)}$ for some constants M and β.

Define, respectively, $F : [0,1] \times X \times X \to X, h : [0,1] \times [0,1] \times X \to X$ and $g : C([0,1], X) \to X$ by

$$F\left(t, z(\sigma(t)), \int\limits_0^t h(t,s,z(\sigma(s)))ds\right)(x) = \frac{z^2(\sin t, x)}{(1+t)(1+t^2)} + \int\limits_0^t \frac{e^{z(\sin s, x)}}{(1+t)(1+t^2)(1+s)^2}ds,$$

$$\int_0^t h(t,s,z(\sigma(s)))(x)ds = \int_0^t \frac{e^{z(\sin s,x)}}{(1+t)(1+t^2)^2(1+s)^2}ds,$$

and

$$g(z)(x) = \sum_{i=1}^{p} c_i \sqrt[3]{z(t_i, x)}, \quad z \in C([0,1], X).$$

Moreover, we have

$$\|F(t,z,y)\| \le \frac{1}{(1+t)(1+t^2)}[\|z\|^2 + \|y\|],$$

where

$$y = \int_0^t \frac{e^{z(\sin s,x)}}{(1+t^2)(1+s)^2}ds,$$

and

$$\|h(t,s,z)\| \le \frac{1}{(1+t^2)(1+s)^2}\exp(\|z\|).$$

It is easy to see that with these choices, the assumptions (H2)–(H8) of Theorem 3.3 are satisfied. Let $\sigma(t) = \sin t$, and hence by Theorem 3.3, we deduce that the nonlocal Cauchy problem (4.1) has a mild solution on $[0,1]$.

5. APPLICATION

This section is devoted to an application of the argument used in previous sections to the controllability of a nonlinear evolution integrodifferential system with nonlocal initial condition in a Banach space X. More precisely, we consider the following problem:

$$\begin{aligned} &x'(t) \in A(t)\left[x(t) + \int_0^t H(t,s)x(s)ds\right] + \\ &\qquad + F\left(t, x(\sigma_1(t)), \dots, x(\sigma_n(t)), \int_0^t h(t,s,x(\sigma_{n+1}(t)))ds\right) + Bu(t), \quad t \in J, \\ &x(0) + g(x) = x_0, \end{aligned} \tag{5.1}$$

where $A(t)$, F and g are as in Section 3. Also, the control function u belongs to the spaces $L^2(J,U)$, a Banach spaces of admissible control functions with U, a Banach space. Further, B is a bounded linear operator from U to X. Recently, the problems of the controllability of differential systems and integrodifferential systems in Banach spaces were considered by many researchers, see for instance [3,6,13] and the references therein. In the case of the nonlocal condition, the semilinear evolution inclusions has been studied by Benchohra *et al.* [7], Guo *et al.* [17], Li and Xue [19].

Definition 5.1. A continuous function $x(\cdot) : J \to X$ is said to be a mild solution to the problem (5.1) if for all $x_0 \in X$, it satisfies the following integral inclusion

$$\begin{aligned} x(t) \in\ & R(t,0)[x_0 - g(x)]+ \\ & + \int_0^t R(t,s)F\left(s, x(\sigma_1(s)), \ldots, x(\sigma_n(s)), \int_0^s h(s,\tau,x(\sigma_{n+1}(\tau)))d\tau\right)ds+ \\ & + \int_0^t R(t,s)Bu(s)ds. \end{aligned} \tag{5.2}$$

Definition 5.2. The system (5.1) is said to be controllable on the interval J if for every $x_0, x_1 \in X$, there exists a control $u \in L^2(J,U)$ such that the mild solution $x(t)$ of system (5.1) satisfies $x(b) + g(x) = x_1$.

We give the following assumptions:

(B1) The linear operator $W : L^2(J,U) \to X$ defined by

$$Wu = \int_0^b R(b,s)Bu(s)ds$$

has an induced inverse operator W^{-1} which takes values in $L^2(J,U) \setminus KerW$ and there exists positive constants M_1 such that $|BW^{-1}| \le M_1$.

(B2) The constants M, M_1, c, b, β satisfy the inequality

$$M_0[M_0 + M_1(1 + Me^{\beta b})bN_0]c < 1, \tag{5.3}$$

where $M_0 = M\max\{1, e^{\beta b}\}$, $N_0 = \max\{1, e^{-\beta b}\}$.

Remark 5.3. The construction of the operator W and its inverse is studied by Quinn and Carmichael in [28].

Theorem 5.4. *Assume that hypotheses* (H1)–(H8)(i), (3.4), (B1) *and* (B2) *are satisfied. Then the system* (5.1) *is controllable on* J.

Proof. Using hypothesis (B1) for each arbitrary function $x(\cdot)$ define the control

$$u_x(t) = W^{-1}\left[x_1 - g(x) - R(b,0)(x_0 - g(x)) - \int_0^b R(b,s)f(s)ds\right](t)$$

for some $f \in S_{F,x}$. It shall be shown that when using this control the operator $P : C(J,X) \to \mathcal{P}(X)$ defined by

$$P(x) := \Big\{\rho \in C(J,X) : \rho(t) = R(t,0)[x_0 - g(x)] + \int_0^t R(t,s)f(s)ds + \\ + \int_0^t R(t,\theta)BW^{-1}\Big[x_1 - g(x) - R(b,0)(x_0 - g(x)) - \\ - \int_0^b R(b,s)f(s)ds\Big](\theta)d\theta, f \in S_{F,x}\Big\}$$

has a fixed point, and then $x(\cdot)$ is a mild solution of systems (5.1). Indeed, it is easy to verify that

$$x_1 - g(x) \in (Px)(b),$$

which means that the system is controllable. The remaining part of the proof is similar to Theorem 3.3, the operator P has a fixed point which is a mild solution of the problem (5.1). Hence, the system (5.1) is controllable on the interval J. □

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