TREES WHOSE 2-DOMINATION SUBDIVISION NUMBER IS 2

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Abstract. A set S of vertices in a graph $G = (V, E)$ is a 2-dominating set if every vertex of $V \setminus S$ is adjacent to at least two vertices of S. The 2-domination number of a graph G, denoted by $\gamma_2(G)$, is the minimum size of a 2-dominating set of G. The 2-domination subdivision number $\mathrm{sd}_{\gamma_2}(G)$ is the minimum number of edges that must be subdivided (each edge in G can be subdivided at most once) in order to increase the 2-domination number. The authors have recently proved that for any tree T of order at least 3, $1 \leq \mathrm{sd}_{\gamma_2}(T) \leq 2$. In this paper we provide a constructive characterization of the trees whose 2-domination subdivision number is 2.

Keywords: 2-dominating set, 2-domination number, 2-domination subdivision number.

Mathematics Subject Classification: 05C69.

1. INTRODUCTION

In this paper, G is a simple graph with vertex set $V(G)$ and edge set $E(G)$ (briefly V and E). For every vertex $v \in V$, the *open neighborhood* $N(v)$ is the set $\{u \in V(G) \mid uv \in E(G)\}$ and the *closed neighborhood* of v is the set $N[v] = N(v) \cup \{v\}$. Similarly, the *open neighborhood* of a set $S \subseteq V$ is the set $N(S) = \cup_{v \in S} N(v)$, and the *closed neighborhood* of S is the set $N[S] = N(S) \cup S$. A *leaf* of a graph G is a vertex of degree 1, while a *support vertex* of G is a vertex adjacent to a leaf. A support vertex is *strong* if it is adjacent to at least two leaves. For a vertex v in a rooted tree T, let $D(v)$ denote the set of descendants of v and $D[v] = D(v) \cup \{v\}$. The *maximal subtree* at v is the subtree of T induced by $D[v]$, and is denoted by T_v.

A 2-*dominating set* of a graph $G = (V, E)$ is a subset S of vertices where each vertex in $V \setminus S$ is adjacent to at least two vertices of S. The 2-*domination number* of a graph G, denoted by $\gamma_2(G)$, is the minimum size of a 2-dominating set of G. A $\gamma_2(G)$-*set* is a 2-dominating set of G with size $\gamma_2(G)$. The 2-domination numbers have been studied by several authors (see for example [6, 7, 13, 15]).

The 2-*domination subdivision number* $\mathrm{sd}_{\gamma_2}(G)$ of a graph G is the minimum number of edges that must be subdivided (where each edge in G can be subdivided at most once) in order to increase the 2-domination number of G. It is easy to see that [4] the 2-domination number of a graph cannot decrease when an edge of that graph is subdivided. For a more thorough treatment of domination parameters and for terminology not presented here see [12, 16].

Atapour *et al.* [4] showed the following result.

Theorem 1.1. *For any tree T of order $n \geq 3$, $1 \leq \mathrm{sd}_{\gamma_2}(T) \leq 2$.*

Hence, trees can be classified as Class 1 or Class 2 depending on whether their 2-domination subdivision numbers are 1 or 2, respectively. In this paper we give a constructive characterization of trees in Class 2. For recent results on the topic "constructive characterization of graphs" the reader may consult [1–3, 9, 11, 14].

We make use of the following observations in this paper.

Theorem 1.2 ([7]). *Every 2-dominating set of a graph G contains every leaf.*

Observation 1.3 ([7]). *Let T be a tree obtained from a nontrivial tree T' by adding a star $K_{1,p}$ with the center vertex v attached by an edge vw at a vertex w of T'. Then $\gamma_2(T') + p \leq \gamma_2(T)$, with equality if $p \geq 2$ or w is a leaf in T'.*

2. TREES WHOSE 2-DOMINATION SUBDIVISION NUMBER IS 2

In this section we provide a constructive characterization of all trees in Class 2. For this purpose, we describe a procedure to build a family $\mathcal{F}$ of labeled trees that are in Class 2 as follows. The label of a vertex is also called its *status* and denoted $sta(v)$. A labeled P_4 is a P_4 where the two leaves have status A and the other two vertices have status B and status C, respectively. Let $\mathcal{F}$ be the family of labeled trees that: A labeled P_4 is a tree in $\mathcal{F}$ and if T is a tree in $\mathcal{F}$, then the tree T' obtained from T by the following five operations which extend the tree T by attaching a tree to a vertex $y \in V(T)$, called an attacher, is also a tree in $\mathcal{F}$.

Operation $\mathfrak{T}_1$. If $sta(y) = B$ (respectively, C) and y is a support vertex, then $\mathfrak{T}_1$ adds a vertex x and an edge xy to T with $sta(x) = A$. Moreover, if $\deg(y) = 2$ and y is adjacent to a vertex z of status C (respectively, B), then this operation changes the status of z to C' (respectively, B').

Operation $\mathfrak{T}_2$. If $sta(y) = B$ (respectively, C) and y is adjacent to a support vertex z with $\deg(z) = 2$ and $sta(z) = C$ (respectively, B), then $\mathfrak{T}_2$ adds a vertex x and an edge xy to T with $sta(x) = A$. Moreover, this operation changes the status of z to C' (respectively, B').

Operation $\mathfrak{T}_3$. If $sta(y) = A, A', B'$ or C', then $\mathfrak{T}_3$ adds a star $K_{1,2}$ with center x and two leaves x_1, x_2 and an edge xy to T with $sta(x) = F$ and $sta(x_1) = sta(x_2) = A$. Moreover, this operation changes the status of y from A to A'.

Operation $\mathfrak{T}_4$. If $sta(y) = A$, then we have three cases:

Case 1. y is adjacent to a vertex z of status B or B'. Then $\mathfrak{T}_4$ adds a path yxu to T with $sta(x) = B$, $sta(u) = A$ and changes the status of y from A to C.

Case 2. y is adjacent to a vertex z of status C or C'. Then $\mathfrak{T}_4$ adds a path yxu to T with $sta(x) = C$, $sta(u) = A$ and changes the status of y from A to B.
Case 3. y is adjacent to a vertex z of status F. Then $\mathfrak{T}_4$ adds a path yxu to T with $sta(x) = C$, $sta(u) = A$ and changes the status of y from A to B.
Operation $\mathfrak{T}_5$. If $sta(y) = F$, then $\mathfrak{T}_5$ adds a vertex x and an edge xy to the tree T with $sta(x) = A$.

The five operations are shown in Figure 1. Note that operation 3 adds two leaves and all the other operations add one leaf to tree T.

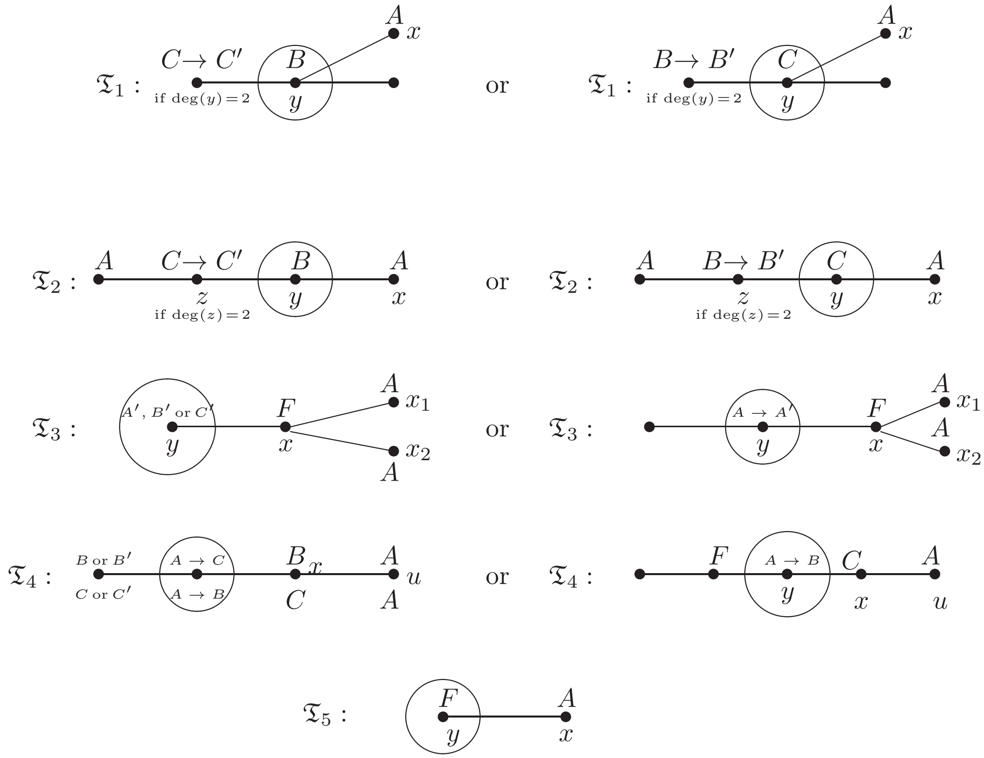

Fig. 1. The five operations

The family $\mathcal{F}$

If $T \in \mathcal{F}$, we let $A(T)$, $B(T)$, $C(T)$, $F(T)$, $A'(T)$, $B'(T)$ and $C'(T)$ be the set of vertices of status A, B, C, F, A', B', and C', respectively, in T. The following observation comes from the way in which each tree in the family $\mathcal{F}$ is constructed.

Observation 2.1. *Let $T \in \mathcal{F}$ and $v \in V(T)$.*

1. *The set of vertices with status A is the set of leaves of tree T.*

2. *If v is a support vertex, then $sta(v) = B, C, F, B'$ or C'.*
3. *If $sta(v) = B$ or B', then v has at least one neighbor y of status C or C' and $N(v) - \{y\} \subset A(T) \cup A'(T) \cup C(T) \cup C'(T) \cup F(T)$. Thus $A(T) \cup A'(T) \cup C(T) \cup C'(T) \cup F(T)$ is a 2-dominating set for T.*
4. *If $sta(v) = C$, C' or F, then v has at least two neighbors of status A, A', B* or *B'. Thus $A(T) \cup A'(T) \cup B(T) \cup B'(T)$ is a 2-dominating set for T.*

We proceed with the following two propositions.

Proposition 2.2. 1. *Let T' be a tree of order at least 3 and let y be a leaf of T'. Let T be a tree obtained from T' by adding a path yuv to T'. Then $\gamma_2(T) = \gamma_2(T') + 1$. Moreover, $\mathrm{sd}_{\gamma_2}(T) \leq \mathrm{sd}_{\gamma_2}(T')$.*

2. *Let T' be a tree of order at least 3 and let y be a strong support vertex of T'. Let T be a tree obtained from T' by adding a pendant edge yw. Then $\gamma_2(T) = \gamma_2(T') + 1$. Moreover, $\mathrm{sd}_{\gamma_2}(T) \leq \mathrm{sd}_{\gamma_2}(T')$.*
3. *Let T' be a tree of order at least 3 and let y be a leaf of T'. Let T be a tree obtained from T' by adding a path yuv to T' and $t(\geq 1)$ pendant edges at y. Then $\gamma_2(T) = \gamma_2(T') + t + 1$. Moreover, $\mathrm{sd}_{\gamma_2}(T) \leq \mathrm{sd}_{\gamma_2}(T')$.*

Proof. (1) By Observation 1.3, $\gamma_2(T) = \gamma_2(T') + 1$. Let F be a set of edges in T' where subdividing the edges in F increases the 2-domination number of T'. Let T_1 and T_2 be the trees obtained from T' and T, respectively, by subdividing the edges in F. Then y is a leaf in T_1 and T_2 is obtained from T_1 by adding a path yuv to T_1. Thus

$$\gamma_2(T_2) = \gamma_2(T_1) + 1 > \gamma_2(T') + 1 = \gamma_2(T).$$

It follows that, $\mathrm{sd}_{\gamma_2}(T) \leq \mathrm{sd}_{\gamma_2}(T')$.

(2) Let u, v be the two leaves of T' adjacent to y in T'. Then u, v, w are leaves in T. It is easy to see that for every $\gamma_2(T')$-set S, $S \cup \{w\}$ is a 2-dominating set of T. It follows that $\gamma_2(T) \leq \gamma_2(T') + 1$. Now if S' is a $\gamma_2(T)$-set, then $\{u, v, w\} \subseteq S'$. Hence, $S' - \{w\}$ is a 2-dominating set of T'. Thus $\gamma_2(T) = \gamma_2(T') + 1$.

Let F be a set of edges in T' where subdividing the edges in F increases the 2-domination number of T'. Let T_1 and T_2 be the trees obtained from T' and T, respectively, by subdividing the edges in F. Then T_2 is obtained from T_1 by adding the pendant edge yw. If $F \cap \{yu, yv\} = \emptyset$, then, as before, we have $\gamma_2(T_2) = \gamma_2(T_1) + 1$ and so

$$\gamma_2(T_2) = \gamma_2(T_1) + 1 > \gamma_2(T') + 1 = \gamma_2(T).$$

Now suppose that $|F \cap \{yu, yv\}| \geq 1$. We may assume the edge yu is subdivided by inserting a vertex x. Obviously, for every $\gamma_2(T_1)$-set S, $S \cup \{w\}$ is a 2-dominating set of T and so $\gamma_2(T_2) \leq \gamma_2(T_1) + 1$. Now if D is a $\gamma_2(T_2)$-set, then by Theorem 1.2, $w \in D$ and to dominate x twice we must have $x \in D$ or $y \in D$. In each case $(D - \{x\}) \cup \{y\}$ is a 2-dominating set for T_1. It follows that $\gamma_2(T_2) = \gamma_2(T_1) + 1$. As before, we have

$$\gamma_2(T_2) = \gamma_2(T_1) + 1 > \gamma_2(T') + 1 = \gamma_2(T).$$

It follows that, $\mathrm{sd}_{\gamma_2}(T) \leq \mathrm{sd}_{\gamma_2}(T')$.

(3) The proof is similar to (1) and (2) and therefore omitted. □

Proposition 2.3. *Let T be a tree obtained from a tree T' of order at least 3 by attaching a star $K_{1,t}$ $(t \geq 2)$ with center x and joining x to a vertex y of T'. Then $\gamma_2(T) = \gamma_2(T') + t$. Moreover,* $\mathrm{sd}_{\gamma_2}(T) \leq \mathrm{sd}_{\gamma_2}(T')$.

Proof. By Observation 1.3, $\gamma_2(T) = \gamma_2(T') + t$. An argument similar to that described in Proposition 2.2 (Part 1) shows that $\mathrm{sd}_{\gamma_2}(T) \leq \mathrm{sd}_{\gamma_2}(T')$. □

Reordering a set of operations with respect to a subset of $\{\mathfrak{T}_i\}_{i=1}^5$

Let T be a tree obtained from a labeled P_4 by successive operations $\mathfrak{T}^1, \ldots, \mathfrak{T}^m$, where $\mathfrak{T}^i \in \{\mathfrak{T}_1, \mathfrak{T}_2, \mathfrak{T}_3, \mathfrak{T}_4, \mathfrak{T}_5\}$ for $1 \leq i \leq m$. Let $J \subseteq \{1, 2, 3, 4, 5\}$ and $\mathfrak{T}_j \in \{\mathfrak{T}_1, \mathfrak{T}_2, \mathfrak{T}_3, \mathfrak{T}_4, \mathfrak{T}_5\}$ for $j \in J$. The following algorithm reorders operations $\mathfrak{T}^1, \ldots, \mathfrak{T}^m$ with respect to $\mathfrak{T}_j$, $j \in J$. It is easy to see that if we apply operations $\mathfrak{T}^i$, $1 \leq i \leq m$ on a labeled P_4, according to the new ordering, we obtain T.

Algorithm

1. Set $k = 0$.
2. Add one to k. If $k > m$, stop.
3. If $\mathfrak{T}^k \notin \{\mathfrak{T}_j \mid j \in J\}$, go to Step 2. If $\mathfrak{T}^k = \mathfrak{T}_j$ for some $j \in J$, proceed as follows. Find the smallest $\ell \in \{1, 2, \ldots, k-1\}$ such that applying $\mathfrak{T}_j$ before $\mathfrak{T}^\ell$ does not lead to a different tree from T. If such an ℓ does not exist, go to Step 2, otherwise apply $\mathfrak{T}_j$ before $\mathfrak{T}^\ell$.

Note that for given successive operations $\mathfrak{T}^1, \ldots, \mathfrak{T}^m$ there exists a unique reordering with respect to a given subset of $\{\mathfrak{T}_1, \mathfrak{T}_2, \mathfrak{T}_3, \mathfrak{T}_4, \mathfrak{T}_5\}$.

Example 2.4. Let T (Figure 2) be obtained by applying the sequence $\mathfrak{T}_3$, $\mathfrak{T}_5$, $\mathfrak{T}_1$, $\mathfrak{T}_4$, $\mathfrak{T}_1$, $\mathfrak{T}_4$, $\mathfrak{T}_3$, $\mathfrak{T}_5$, $\mathfrak{T}_4$ on the initial path $x_1x_2x_3x_4$. We see that $\mathfrak{T}_3$ adds the star with center x_5 and the leaves x_6 and x_7 to x_4 (Figure 3), $\mathfrak{T}_5$ adds x_8 to x_5 (Figure 4), $\mathfrak{T}_1$ adds x_9 to x_2, $\mathfrak{T}_4$ adds $x_{10}x_{11}$ to x_8, $\mathfrak{T}_1$ adds x_{12} to x_{10} (Figure 5), $\mathfrak{T}_4$ adds $x_{13}x_{14}$ to x_{11}, $\mathfrak{T}_3$ adds the star with center x_{15} and the leaves x_{16} and x_{17} to x_3 (Figure 6), $\mathfrak{T}_5$ adds x_{18} to x_{15} and $\mathfrak{T}_4$ adds the path $x_{19}x_{20}$ to x_{17} (Figure 7). Then T is in Figure 2.

In what follows, we step by step show that how one can find the reordering of the operations $\mathfrak{T}_3, \mathfrak{T}_5, \mathfrak{T}_1, \mathfrak{T}_4, \mathfrak{T}_1, \mathfrak{T}_4, \mathfrak{T}_3, \mathfrak{T}_5, \mathfrak{T}_4$ with respect to $\{\mathfrak{T}_1, \mathfrak{T}_3, \mathfrak{T}_5\}$. The new ordering will be $\mathfrak{T}_3, \mathfrak{T}_5, \mathfrak{T}_1, \mathfrak{T}_3, \mathfrak{T}_5, \mathfrak{T}_4, \mathfrak{T}_1, \mathfrak{T}_4, \mathfrak{T}_4$.

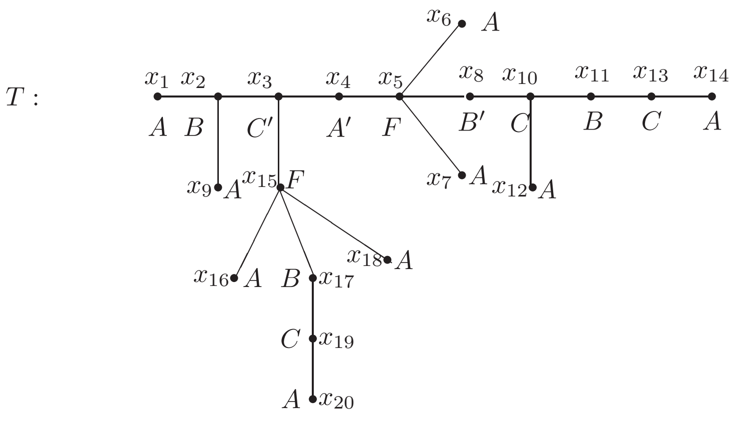

Fig. 2

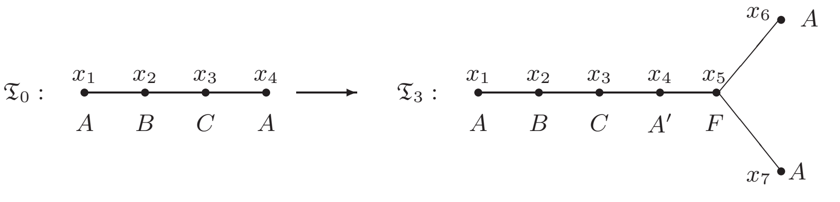

Fig. 3

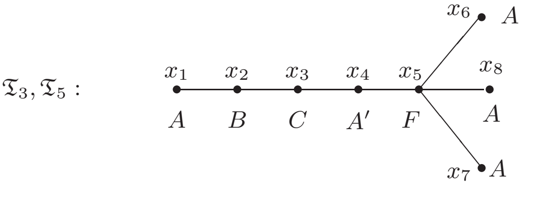

Fig. 4

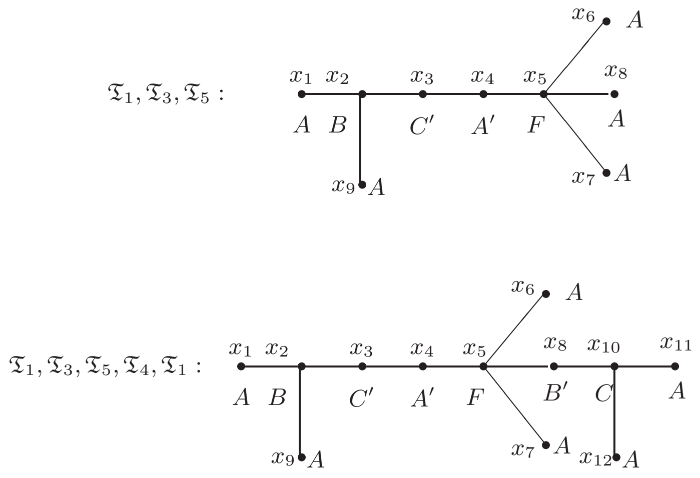

Fig. 5

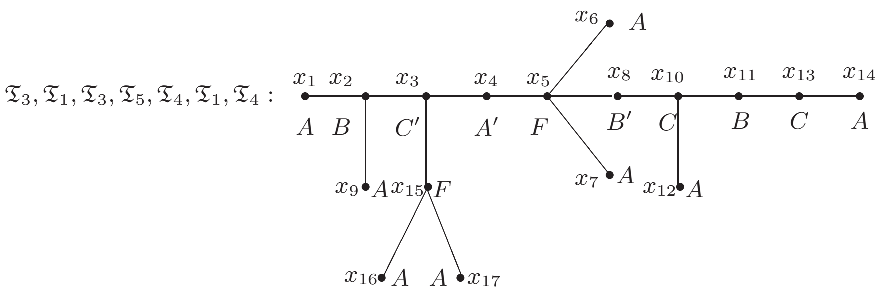

Fig. 6

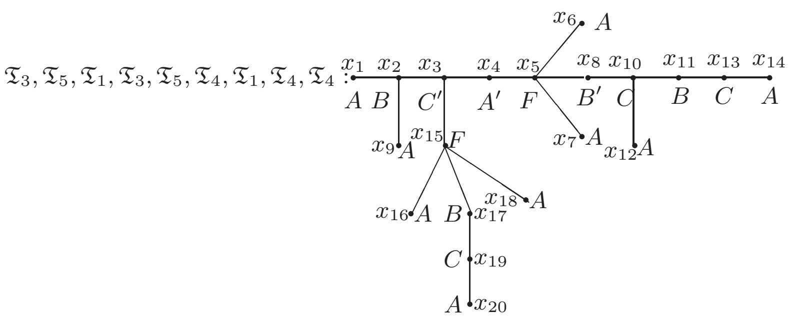

Fig. 7

In order to show that each tree in the family $\mathcal{F}$ is in Class 2, we first present three lemmas.

Lemma 2.5. *Let $T \in \mathcal{F}$ be obtained from a labeled P_4 by successive operations $\mathfrak{T}^1, \ldots, \mathfrak{T}^m$, where $\mathfrak{T}^i \in \{\mathfrak{T}_1, \mathfrak{T}_2, \mathfrak{T}_3, \mathfrak{T}_4, \mathfrak{T}_5\}$ if $m \geq 1$ and $T = P_4$ if $m = 0$. Then $A(T) \cup A'(T) \cup B(T) \cup B'(T)$ is a 2-dominating set of T and $\gamma_2(T) = m + k + 3$, where k is the number of operations of type $\mathfrak{T}_3$.*

Proof. By Part (4) of Observation 2.1, the set $A(T) \cup A'(T) \cup B(T) \cup B'(T)$ is a 2-dominating set of T. This implies that $\gamma_2(T) \leq m + k + 3$. The proof of $\gamma_2(T) = m+k+3$ is by induction on m. If $m = 0$, then clearly the statement is true. Let $m \geq 1$ and that the statement holds for all trees which are obtained from P_4 by applying $m - 1$ operations $\mathfrak{T} \in \{\mathfrak{T}_1, \mathfrak{T}_2, \mathfrak{T}_3, \mathfrak{T}_4, \mathfrak{T}_5\}$. Reorder the operations $\{\mathfrak{T}^1, \mathfrak{T}^2, \ldots, \mathfrak{T}^m\}$ with respect to $\{\mathfrak{T}_1, \mathfrak{T}_2, \mathfrak{T}_5\}$. Let T_{m-1} be the tree obtained from P_4 by the first $m-1$ operations $\mathfrak{T}^1, \ldots, \mathfrak{T}^{m-1}$. If $\mathfrak{T}^m = \mathfrak{T}_3$, then T has been obtained from T_{m-1} by adding a star $K_{1,2}$ with center x and two leaves x_1, x_2 and an edge xy to T. By the inductive hypothesis, $\gamma_2(T_{m-1}) = (m - 1) + (k - 1) + 3 = m + k + 1$ and the result follows by Proposition 2.3. If $\mathfrak{T}^m = \mathfrak{T}_5$, then T has been obtained from T_{m-1} by adding a vertex x and an edge xy to the tree T_{m-1} where $sta_{T_{m-1}}(y) = F$. Then, by the choice of reordering, y is a strong support vertex in T_{m-1}. By the inductive hypothesis, $\gamma_2(T_{m-1}) = (m - 1) + k + 3 = m + k + 2$ and the result follows by Proposition 2.2 (Part 2). If $\mathfrak{T}^m = \mathfrak{T}_4$, then the result follows by the inductive hypothesis and Proposition 2.2 (Part 1). Now consider the two remaining cases.

Case 1. $\mathfrak{T}^m = \mathfrak{T}_1$. Then T has been obtained from T_{m-1} by adding a vertex x and an edge xy, where y is a support vertex of T_{m-1}. Suppose that w is a leaf adjacent to y and z is a vertex of status B, C, C' or B' adjacent to y by Observation 2.1, Parts (2) and (3). First assume y is in the original P_4. Then, by the choice of reordering, $\mathfrak{T}^1 = \mathfrak{T}^2 = \ldots = \mathfrak{T}^m = \mathfrak{T}_1$ and each operation adds a pendant edge at y. Therefore $\deg(z) = 2$. For any $\gamma_2(T_{m-1})$-set S', $S' \cup \{x\}$ is a 2-dominating set of T and so $\gamma_2(T) \leq \gamma_2(T_{m-1}) + 1$. On the other hand, if S is a $\gamma_2(T)$-set, then clearly $x, w \in S$ and $|S \cap \{y, z\}| \geq 1$ since $\deg(z) = 2$. Then $S - \{x\}$ is a 2-dominating set of T_{m-1}. This implies that $\gamma_2(T_{m-1}) \leq \gamma_2(T) - 1$ and so $\gamma_2(T_{m-1}) + 1 = \gamma_2(T)$. Now the result follows by the inductive hypothesis.

Now assume y is not in the original P_4. By the choice of reordering, we may assume for some positive integer s, $\mathfrak{T}^m = \ldots = \mathfrak{T}^{s+1} = \mathfrak{T}_1$ and each operation adds a pendant edge at y and $\mathfrak{T}^s = \mathfrak{T}_4$ which adds the path zyw. Therefore, z is a leaf in T_{s-1} and so $sta_{T_{s-1}}(z) = A$ and $\deg_T(z) = 2$. By Proposition 2.3, $\gamma_2(T_{s-1}) + (m - s) + 1 = \gamma_2(T)$. Now the result follows by the inductive hypothesis.

Case 2. $\mathfrak{T}^m = \mathfrak{T}_2$. Then T has been obtained from T_{m-1} by adding a vertex x and an edge xy, where y is adjacent to a support vertex z of T_{m-1} with $\deg(z) = 2$. For any $\gamma_2(T_{m-1})$-set S', $S' \cup \{x\}$ is a 2-dominating set of T and so $\gamma_2(T) \leq \gamma_2(T_{m-1}) + 1$. On the other hand, if S is a $\gamma_2(T)$-set, then clearly $x, w \in S$ and $|S \cap \{y, z\}| \geq 1$ since $\deg(z) = 2$. Then $S - \{x\}$ is a 2-dominating set of T_{m-1}. This implies that $\gamma_2(T_{m-1}) \leq \gamma_2(T) - 1$ and so $\gamma_2(T_{m-1}) + 1 = \gamma_2(T)$. Now the result follows by the inductive hypothesis. □

Lemma 2.6. *Let $T \in \mathcal{F}$ be obtained from a labeled P_4 by successive operations $\mathfrak{T}^1, \ldots, \mathfrak{T}^m$, where $\mathfrak{T}^i \in \{\mathfrak{T}_1, \mathfrak{T}_2, \mathfrak{T}_3, \mathfrak{T}_4, \mathfrak{T}_5\}$ if $m \geq 1$ and $T = P_4$ if $m = 0$. Then:*

1. *for every $v \in V(T)$, there exists a $\gamma_2(T)$-set containing v,*
2. *if $v \in A(T)$, then there is a $\gamma_2(T)$-set S containing v and its support vertex. Therefore, $S - \{v\}$ is a 2-dominating set of $T - \{v\}$.*

Proof. The proof is by induction on m. If $m = 0$, then clearly the statements are true. Let $m \geq 1$ and the statements hold for all trees which are obtained from a labeled P_4 by applying at most $m-1$ operations $\mathfrak{T} \in \{\mathfrak{T}_1, \mathfrak{T}_2, \mathfrak{T}_3, \mathfrak{T}_4, \mathfrak{T}_5\}$. Let T_{m-1} be the tree obtained from P_4 by the first $m-1$ operations $\mathfrak{T}^1, \ldots, \mathfrak{T}^{m-1}$. Reorder the operations $\{\mathfrak{T}^1, \mathfrak{T}^2, \ldots, \mathfrak{T}^m\}$ with respect to $\{\mathfrak{T}_3\}$.

(1) Since by Lemma 2.5, $A(T) \cup A'(T) \cup B(T) \cup B'(T)$ is a $\gamma_2(T)$-set, we assume that $v \in C(T) \cup C'(T) \cup F(T)$. We consider three cases.

Case 1. $\mathfrak{T}^m = \mathfrak{T}_1, \mathfrak{T}_2$ or $\mathfrak{T}_5$. Then T is obtained from T_{m-1} by adding a vertex x and an edge xy, where $y \in B(T_{m-1}) \cup C(T_{m-1}) \cup F(T_{m-1})$. Since $C(T) \cup C'(T) \cup F(T) = C(T_{m-1}) \cup C'(T_{m-1}) \cup F(T_{m-1})$, by the inductive hypothesis v is contained in some $\gamma_2(T_{m-1})$-set S. Now $S \cup \{x\}$ is a $\gamma_2(T)$-set containing v by Lemma 2.5.

Case 2. $\mathfrak{T}^m = \mathfrak{T}_3$. Then T is obtained from T_{m-1} by adding a star $K_{1,2}$ with center x and two leaves x_1, x_2 and an edge xy, where $y \in A(T_{m-1}) \cup A'(T_{m-1}) \cup B'(T_{m-1}) \cup C'(T_{m-1})$. We have $C(T) \cup C'(T) \cup F(T) = (C(T_{m-1}) \cup C'(T_{m-1}) \cup F(T_{m-1})) \cup \{x\}$. If $v \in V(T_{m-1})$, then by the inductive hypothesis there is a $\gamma_2(T_{m-1})$-set S containing v and $S \cup \{x_1, x_2\}$ is a $\gamma_2(T)$-set by Lemma 2.5. Let $v = x$. By the choice of reordering, for some integer $0 \leq s \leq m-1$, each of $\mathfrak{T}^m, \mathfrak{T}^{m-1}, \cdots, \mathfrak{T}^{s+1}$ adds a star $K_{1,2}$ and joins its center to y but $\mathfrak{T}^s$ does not add a star $K_{1,2}$ to y. If $s < m-1$, then we may assume $\mathfrak{T}^{m-1}$ adds a star $K_{1,2}$ with center x' and leaves x'_1, x'_2. Obviously, we can rearrange the order of the operations to have $\mathfrak{T}^1, \ldots, \mathfrak{T}^{m-2}$, $\mathfrak{T}^m, \mathfrak{T}^{m-1}$. By the inductive hypothesis, the tree T' obtained from P_4 by the operations $\mathfrak{T}^1, \ldots, \mathfrak{T}^{m-2}$, $\mathfrak{T}^m$ has a $\gamma_2(T')$-set S containing v. Then $S \cup \{x'_1, x'_2\}$ is a $\gamma_2(T)$-set containing v by Lemma 2.5. Now we assume $s = m-1$. Let first $\mathrm{sta}(y) = B'$ or C'. Then, by the choice of reordering, we may assume $\mathfrak{T}^s \in \{\mathfrak{T}_1, \mathfrak{T}_2\}$. We consider two subcases.

Subcase 2.1. $\mathfrak{T}^{m-1} = \mathfrak{T}_1$. This forces that y is adjacent to a strong support vertex z with status B or C and $\deg(z) = 3$. By Lemma 2.5 and the inductive hypothesis, z is contained in a $\gamma_2(T_{m-1})$-set S. Now obviously $(S \setminus \{z\}) \cup \{x, x_1, x_2\}$ is a $\gamma_2(T)$-set containing v.

Subcase 2.2. $\mathfrak{T}^{m-1} = \mathfrak{T}_2$. Then T_{m-1} is obtained from T_{m-2} by adding a vertex u and an edge uz, where z is a vertex of status B or C adjacent to the support vertex y of status C or B and degree 2 in T_{m-2}. Thus we have $\deg_{T_{m-1}}(z) \geq 3$, $\mathrm{sta}_{T_{m-1}}(z) = B$ or C and $\mathrm{sta}_{T_{m-1}}(y) = C'$ or B'. Let z' be a vertex adjacent to z other than y and u. By the inductive hypothesis, z' is contained in a $\gamma_2(T_{m-1})$-set say S. Then we have $z \in S$ or $y \in S$. By Lemma 2.5, $(S \setminus \{z, y\}) \cup \{x, x_1, x_2\}$ is a $\gamma_2(T)$-set containing v.

Now let $sta(y) = A$. Then y is a leaf in T_{m-1}, and by the inductive hypotheses there is a $\gamma_2(T_{m-1})$-set S containing y and its support vertex and so $(S \setminus \{y\}) \cup \{x, x_1, x_2\}$ is a $\gamma_2(T)$-set containing v.

Finally, let $sta(y) = A'$. Then $\mathfrak{T}^{m-1}$ adds a star $K_{1,2}$ with center x' and leaves x'_1, x'_2 and changes the status of y from A to A'. Thus y is a leaf in T_{m-2}, and by the inductive hypothesis there is a $\gamma_2(T_{m-2})$-set S containing y and its support vertex w. Now obviously $(S \setminus \{y\}) \cup \{x'_1, x'_2, x, x_1, x_2\}$ is a $\gamma_2(T)$-set containing v.

Case 3. $\mathfrak{T}^m = \mathfrak{T}_4$. Then T is obtained from T_{m-1} by adding a path yxu to T_{m-1}, where $y \in A(T_{m-1})$. Thus y is a leaf in T_{m-1}. Suppose that z is the support vertex of y in T_{m-1}. If $v \in T_{m-1}$, then by the inductive hypothesis v is contained in some $\gamma_2(T_{m-2})$-set S and $S \cup \{u\}$ is a $\gamma_2(T)$-set by Lemma 2.5. Now let $v = x$. By the inductive hypothesis, there is a $\gamma_2(T_{m-1})$-set S containing y and its support vertex and obviously $(S - \{y\}) \cup \{x, u\}$ is a $\gamma_2(T)$-set containing v.

(2) Let u be the support vertex of v. Then by Part (2) of Observation 2.1, $sta(u) = B, C, B', C'$, or F. Now the result follows by Lemma 2.5, Part (1) of this theorem and the fact that each $\gamma_2(T)$-set contains all leaves □

Lemma 2.7. *Let $T \in \mathcal{F}$ and let T^* be a tree obtained from T by subdividing an edge of T. Then $\gamma_2(T^*) = \gamma_2(T)$.*

Proof. Let $T \in \mathcal{F}$. First note that $\gamma_2(T^*) \geq \gamma_2(T)$ and that any 2-dominating set of T^* of size $\gamma_2(T)$ is a $\gamma_2(T^*)$-set. Let $e \in E(T)$ and let T^* be obtained from T by subdividing the edge e with vertex u. Let T be obtained from a labeled P_4 by successive operations $\mathfrak{T}^1, \ldots, \mathfrak{T}^m$, respectively, where $\mathfrak{T}^i \in \{\mathfrak{T}_1, \mathfrak{T}_2, \mathfrak{T}_3, \mathfrak{T}_4, \mathfrak{T}_5\}$ for $1 \leq i \leq m$ if $m \geq 1$ and $T = P_4$ if $m = 0$. The proof is by induction on m. If $m = 0$, then clearly the statement is true. Assume $m \geq 1$ and that the statement holds for all trees which are obtained from a labeled P_4 by applying at most $m-1$ operations. Suppose T_{m-1} is a tree obtained by applying the first $m-1$ operations $\mathfrak{T}^1, \ldots, \mathfrak{T}^{m-1}$. When $e \in E(T_{m-1})$, let T^*_{m-1} be obtained from T_{m-1} by subdividing the edge e with vertex u. We consider three cases.

Case 1. $\mathfrak{T}^m = \mathfrak{T}_1, \mathfrak{T}_2$ or $\mathfrak{T}_5$. Then T is obtained from T_{m-1} by attaching the path yx to $y \in B(T_{m-1}) \cup C(T_{m-1}) \cup F(T_{m-1})$. If $e \in E(T_{m-1})$, then by the inductive hypothesis we have

$$\gamma_2(T^*) \leq \gamma_2(T^*_{m-1}) + 1 = \gamma_2(T_{m-1}) + 1 = \gamma_2(T).$$

Let $e = xy$. By Lemmas 2.5 and 2.6, there exists a $\gamma_2(T_{m-1})$-set S containing y. Now $S \cup \{x\}$ is a 2-dominating set of T^* of size $\gamma_2(T_{m-1}) + 1 = \gamma_2(T)$. Hence, $\gamma_2(T^*) = \gamma_2(T)$.

Case 2. $\mathfrak{T}^m = \mathfrak{T}_3$. Then T is obtained from T_{m-1} by attaching a star $K_{1,2}$ with center x and two leaves x_1, x_2 to the attacher $y \in A(T_{m-1}) \cup A'(T_{m-1}) \cup C'(T_{m-1}) \cup B'(T_{m-1})$. If $e \in E(T_{m-1})$, then by Proposition 2.3 and the inductive hypothesis we have

$$\gamma_2(T^*) = \gamma_2(T^*_{m-1}) + 2 = \gamma_2(T_{m-1}) + 2 = \gamma_2(T).$$

Let $e \in E(T) \setminus E(T_{m-1})$. By Lemma 2.6, there is a $\gamma_2(T)$-set S containing x. Now S is a 2-dominating set of T^* of size $\gamma_2(T)$ if $e = xx_1$ or xx_2 and $(S - \{x\}) \cup \{u\}$ is a 2-dominating set for T^* of size $\gamma_2(T)$ if $e = xy$. Recall that u is the subdividing vertex.

Case 3. $\mathfrak{T}^m = \mathfrak{T}_4$. Then T is obtained from T_{m-1} by attaching the path yxw to the attacher $y \in A(T_{m-1})$. If $e \in E(T_{m-1})$, then by Proposition 2.2 and the inductive hypothesis $\gamma_2(T^*) = \gamma_2(T^*_{m-1}) + 1 = \gamma_2(T_{m-1}) + 1 = \gamma_2(T)$. Let $e \notin E(T_{m-1})$. Without loss of generality, we may subdivide $e = yx$ with u. By Lemma 2.6, T_{m-1} has a $\gamma_2(T_{m-1})$-set S containing y and its support vertex. Now $(S - \{y\}) \cup \{u, w\}$ is a $\gamma_2(T^*)$-set of size $\gamma_2(T)$. This completes the proof. $\square$

An immediate consequence of Theorem 1.1 and Lemma 2.7 now follows.

Theorem 2.8. *Each tree in Family $\mathcal{F}$ is in Class 2.*

In order to prove that any tree in Class 2 is indeed in $\mathcal{F}$ we need the following lemma.

Lemma 2.9. *Let $T \in \mathcal{F}$, $v \in B(T) \cup C(T) \cup F(T)$ and let T^* be obtained from T by adding a star $K_{1,2}$ and an edge joining the center of the star to v. Then* $\mathrm{sd}_{\gamma_2}(T^*) = 1$.

Proof. Let $T \in \mathcal{F}$ be obtained from a labeled P_4 by successive operations $\mathfrak{T}^1, \ldots, \mathfrak{T}^m$, where $\mathfrak{T}^i \in \{\mathfrak{T}_1, \mathfrak{T}_2, \mathfrak{T}_3, \mathfrak{T}_4, \mathfrak{T}_5\}$ if $m \geq 1$ and $T = P_4$ if $m = 0$. The proof is by induction on m. If $m = 0$, then clearly the statement is true. Assume $m \geq 1$ and that the statement holds for all trees which are obtained from a labeled P_4 by applying at most $m - 1$ operations. Suppose T_{m-1} is the tree obtained by applying the first $m - 1$ operations $\mathfrak{T}^1, \ldots, \mathfrak{T}^{m-1}$. When $v \in V(T_{m-1})$, let T^*_{m-1} be obtained from T_{m-1} by adding a star $K_{1,2}$ and an edge joining the center of the star to v. Reorder the operations $\{\mathfrak{T}^1, \mathfrak{T}^2, \ldots, \mathfrak{T}^m\}$ with respect to $\{\mathfrak{T}_1, \mathfrak{T}_2, \mathfrak{T}_5\}$. Let z, z_1 and z_2 be the center and leaves of the added star to T, respectively. We consider five cases.

Case 1. $\mathfrak{T}^m = \mathfrak{T}_3$. Then T is obtained from T_{m-1} by adding a star $K_{1,2}$ and an edge joining the center x of the star to $y \in A(T_{m-1}) \cup A'(T_{m-1}) \cup B'(T_{m-1}) \cup C'(T_{m-1})$. If $v \in V(T_{m-1})$, then by the inductive hypothesis $\mathrm{sd}_{\gamma_2}(T^*_{m-1}) = 1$. Since T^* is formed from T^*_{m-1} by adding a star $K_{1,2}$, by Proposition 2.3 we have $\mathrm{sd}_{\gamma_2}(T^*) \leq \mathrm{sd}_{\gamma_2}(T^*_{m-1}) = 1$. Thus by Theorem 1.1, $\mathrm{sd}_{\gamma_2}(T^*) = 1$. If $v = x$, then let T' be obtained from T^* by subdividing the edge xz by inserting a vertex t. Since y and z are strong support vertices, for each $\gamma_2(T^*)$-set S we have $z \notin S$, for otherwise $S - \{z\}$ is a 2-dominating set for T^*, a contradiction. Let D be a $\gamma_2(T')$-set. Then $u \in D$ or $y, z \in D$ and hence $D - \{u\}$ or $D - \{z\}$ is a 2-dominating set for T^*. Therefore $\mathrm{sd}_{\gamma_2}(T^*) \leq 1$ and the result follows by Theorem 1.1.

Case 2. $\mathfrak{T}^m = \mathfrak{T}_4$. Then T is obtained from T_{m-1} by adding a path xw and an edge joining x to $y \in A(T_{m-1})$. First let $v \in V(T_{m-1}) - \{y\}$. Then by the inductive hypothesis $\mathrm{sd}_{\gamma_2}(T^*_{m-1}) = 1$. Assume e is an edge of T^*_{m-1} such that subdividing e increases the 2-domination number. Let T'_{m-1} and T' be obtained from T^*_{m-1} and

T^* by subdividing the edge e, respectively. By Proposition 2.2 (Part (1)), $\gamma_2(T^*) = \gamma_2(T^*_{m-1}) + 1$ and $\gamma_2(T') = \gamma_2(T'_{m-1}) + 1$. Now

$$\gamma_2(T') = \gamma_2(T'_{m-1}) + 1 \geq \gamma_2(T^*_{m-1}) + 2 = \gamma_2(T^*) + 1.$$

Therefore, $\mathrm{sd}_{\gamma_2}(T^*) = 1$ by Theorem 1.1.

Let $v = y$. Obviously, $\deg(x) = 2$. Let T' be obtained from T^* by subdividing the edge yz by inserting a vertex t. Suppose that S is a $\gamma_2(T')$-set. Since $\deg(x) = 2$, $y \in S$ or $x \in S$. We may assume $y \in S$, otherwise $(S - \{x\}) \cup \{y\}$ is a $\gamma_2(T')$-set. Since t is a subdividing vertex, $\deg(t) = 2$. To dominate t we must have $S \cap \{t, z\} \neq \emptyset$. Now obviously $S - \{t, z\}$ is a 2-dominating set for T^* and so $\mathrm{sd}_{\gamma_2}(T^*) = 1$ by Theorem 1.1.

Now let $v = x$. Then $\deg_{T^*}(y) = 2$. Suppose that $w \neq x$ is adjacent to y. Let T' be obtained from T^* by subdividing the edge xz by inserting a vertex t. Suppose that S is a $\gamma_2(T')$-set. Since $\deg_{T'}(y) = 2$, $y \in S$ or $\{x, w\} \subseteq S$. To dominate t we must have $S \cap \{t, z\} \neq \emptyset$. Now obviously $S - \{t, z\}$ is a 2-dominating set for T^* and so $\mathrm{sd}_{\gamma_2}(T^*) = 1$ by Theorem 1.1.

Case 3. $\mathfrak{T}^m = \mathfrak{T}_5$. Then T is obtained from T_{m-1} by adding a vertex x and an edge joining x to $y \in F(T_{m-1})$. By the choice of reordering, we may assume $\mathfrak{T}^m = \ldots = \mathfrak{T}^{k+1} = \mathfrak{T}_5$ and $\mathfrak{T}^k = \mathfrak{T}_3$ which adds a star $K_{1,2}$ with center y. Suppose T_{k-1} is the tree obtained by applying the first $k-1$ operations $\mathfrak{T}^1, \ldots, \mathfrak{T}^{k-1}$. If $v \in V(T_{k-1})$, then Proposition 3 and an argument similar to that described in Case 2 show that the statement is true. If $v = y$, then let T' be obtained from T^* by subdividing the edge vz by inserting a vertex t. Since y and z are strong support vertices, for each $\gamma_2(T^*)$-set S we have $z \notin S$, for otherwise $S - \{z\}$ is a 2-dominating set for T^*, a contradiction. Let D be a $\gamma_2(T')$-set. Then $u \in D$ or $y, z \in D$ and hence $D - \{u\}$ or $D - \{z\}$ is a 2-dominating set for T^*. Therefore $\mathrm{sd}_{\gamma_2}(T^*) \leq 1$ and the result follows by Theorem 1.1.

Case 4. $\mathfrak{T}^m = \mathfrak{T}_1$. Then T is obtained from T_{m-1} by adding a vertex x and an edge joining x to a support vertex $y \in B(T_{m-1}) \cup C(T_{m-1})$. If y belongs to the original P_4, then obviously $\mathfrak{T}^1 = \ldots = \mathfrak{T}^m = \mathfrak{T}_1$ and each operation adds a pendant edge at y. This forces $v = y$ and as Case 3, it is easy to see that subdividing the edge yz increases the 2-domination number. Suppose y is not contained in the original P_4. By the choice of reordering, we may assume $\mathfrak{T}^m = \ldots = \mathfrak{T}^{s+1} = \mathfrak{T}_1$ where each operation adds a pendant edge at y and $\mathfrak{T}^s = \mathfrak{T}_4$ which adds a path yw and an edge joining y to some vertex in T_{s-1}. Suppose T_{s-1} is the tree obtained by applying the first $s-1$ operations $\mathfrak{T}^1, \ldots, \mathfrak{T}^{s-1}$. If $v \in V(T_{s-1})$, then Proposition 2.3 and an argument similar to that described in Case 2 show that the statement is true. If $v = y$ then, as before, we can see that subdividing the edge yz increases the 2-domination number.

Case 5. $\mathfrak{T}^m = \mathfrak{T}_2$. Then T is obtained from T_{m-1} by adding a vertex x and an edge that joins x to a vertex $y \in B(T_{m-1}) \cup C(T_{m-1})$, where y is adjacent to a support vertex z of degree 2 in T_{m-1}. If y belongs to the original P_4, then the result follows as Case 4. Suppose y is not contained in the original P_4. By the choice of reordering, we may assume $\mathfrak{T}^m = \ldots = \mathfrak{T}^{s+1} = \mathfrak{T}_2$ where each operation adds a pendant edge at y and $\mathfrak{T}^s = \mathfrak{T}_4$ which adds the path zw and an edge joining z to y in T_{s-1}.

Suppose T_{s-1} is the tree obtained by applying the first $s-1$ operations $\mathfrak{T}^1, \ldots, \mathfrak{T}^{s-1}$. If $v \in V(T_{s-1})$, then the result follows by Proposition 2.2 (Part (3)) and the inductive hypothesis. Let $v = y$. We show that subdividing the edge zz_1, where z_1 is a leaf at z, increases the 2-domination number. Let T' be obtained from T^* by subdividing the edge zz_1 by inserting a vertex u. Let S be a $\gamma_2(T)$-set. Since $\deg_T(z) = 2$, we may assume $y \in S$. Now to dominate u we must have $u \in S$ or $z \in S$. Then clearly $S - \{u, z\}$ is a 2-dominating set for T^*. It follows that $\mathrm{sd}_{\gamma_2}(T) = 1$. This completes the proof. □

Theorem 2.10. *A tree T of order $n \geq 3$ is in Class 2 if and only if $T \in \mathcal{F}$.*

Proof. By Theorem 2.8, we only need to prove that every tree in Class 2 is in $\mathcal{F}$. We prove this by induction on n. Since $\mathrm{sd}_{\gamma_2}(T) = 2$, we have $n \geq 4$. If $n = 4$, then the only tree T of order 4 and $\mathrm{sd}_{\gamma_2}(T) = 2$ is $P_4 \in \mathcal{F}$. Let $n \geq 5$ and assume the statement holds for every tree in Class 2 of order less than n. Let T be a tree of order n and $\mathrm{sd}_{\gamma_2}(T) = 2$. Assume $P = v_1 v_2 \ldots v_r$ is the longest path in T. Obviously, $\deg(v_1) = \deg(v_r) = 1$ and $r \geq 4$. Suppose T is rooted at v_r.

First let $\deg(v_2) \geq 3$. Then v_2 is a strong support vertex. Let $v_1 = u_1, u_2, \ldots, u_{\deg(v_2)-1}$ be the leaves adjacent to v_2 and $T_1 = T - T_{v_2}$. By Proposition 2.3, $\mathrm{sd}_{\gamma_2}(T_1) = 2$ and by the inductive hypothesis, $T_1 \in \mathcal{F}$. Since $\mathrm{sd}_{\gamma_2}(T) = 2$, by Lemma 2.9, $sta_{T_1}(v_3) = A, A', B'$, or C', and hence T can be obtained from T_1 by applying operation $\mathfrak{T}_3$ once and operation $\mathfrak{T}_5$, $\deg(v_2) - 3$ times.

Now let $\deg(v_2) = 2$. First let $\deg(v_3) = 2$. Then by Proposition 2.2 (Part (1)), $\gamma_2(T) = \gamma_2(T - T_{v_2}) + 1$ and $\mathrm{sd}_{\gamma_2}(T) \leq \mathrm{sd}_{\gamma_2}(T - T_{v_2})$. Therefore $\mathrm{sd}_{\gamma_2}(T - T_{v_2}) = 2$ and by the inductive hypothesis, $T - T_{v_2} \in \mathcal{F}$. Now T can be obtained from $T - T_{v_2}$ by operation $\mathfrak{T}_4$. Now let $\deg(v_3) \geq 3$. First assume that v_3 is adjacent to a support vertex u such that $u \neq v_2$. Let w be a leaf adjacent to u. As before, we may assume that $\deg(u) = 2$. Let T' be obtained from T by subdividing the edge v_3u by inserting a vertex s. For any $\gamma_2(T)$-set S of T, $|S \cap \{v_1, v_2, v_3\}| \geq 2$ and $|S \cap \{s, u, w\}| \geq 2$. Obviously, $(S - \{v_1, v_2, v_3, s, u, w\}) \cup \{v_1, v_3, w\}$ is a 2-dominating set for T with cardinality less than $|S|$. Therefore, $\mathrm{sd}_{\gamma_2}(T) = 1$, a contradiction. Thus v_3 is adjacent to $\deg(v_3) - 2$ leaves. Let $u_1, \ldots, u_{\deg(v_3)-2}$ be the leaves adjacent to v_3. Assume $T' = T - \{u_1, \ldots, u_{\deg(v_3)-2}, v_1, v_2\}$. By Proposition 2.2 (Part 3) $\gamma_2(T) = \gamma_2(T') + \deg(v_3) - 1$ and $\mathrm{sd}_{\gamma_2}(T) \leq \mathrm{sd}_{\gamma_2}(T')$. Since $\mathrm{sd}_{\gamma_2}(T) = 2$, by Theorem 1.1 , $\mathrm{sd}_{\gamma_2}(T') = 2$. Hence, by the inductive hypothesis, $T' \in \mathcal{F}$. Since v_3 is a leaf in T', $sta_{T'}(v_3) = A$ and T can be obtained from T' by applying operation $\mathfrak{T}_4$ once and operations $\mathfrak{T}_1$ or $\mathfrak{T}_2$, $\deg(v_3) - 2$ times. Thus $T \in \mathcal{F}$ and the proof is complete. □

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