Opuscula Mathematica • Vol. 32 • No. 3 • 2012

STEPANOV-LIKE $C^{(n)}$-PSEUDO ALMOST AUTOMORPHY AND APPLICATIONS TO SOME NONAUTONOMOUS HIGHER-ORDER DIFFERENTIAL EQUATIONS

Toka Diagana, Valerie Nelson, and Gaston M. N'Guérékata

Abstract. In this paper we introduce and study a new concept called Stepanov-like $C^{(n)}$-pseudo almost automorphy, which generalizes in a natural fashion both the notions of $C^{(n)}$-pseudo almost periodicity and that of $C^{(n)}$-pseudo almost automorphy recently introduced in the literature by the authors. Basic properties of these new functions are investigated. Furthermore, we study and obtain the existence of $C^{(N+m)}$-pseudo almost automorphic solutions to some nonautonomous higher-order systems of differential equations with Stepanov-like $C^{(m)}$-pseudo almost automorphic coefficients.

Keywords: pseudo almost automorphic $C^{(n)}$-pseudo almost automorphy, Stepanov-like $C^{(n)}$-pseudo almost automorphy, exponential dichotomy.

Mathematics Subject Classification: 35B15, 34D09, 58D25, 42A75, 37L05.

1. INTRODUCTION

A few years ago, following a suggestion by N'Guérékata [39, p. 40], Xiao *et al.* [34,40, 41] introduced the concept of pseudo almost automorphy in the literature, which in fact is a very interesting generalization of the notions of periodicity, almost periodicity, almost automorphy, and that of the pseudo almost periodicity. Since then, the concept of pseudo almost automorphy has generated several developments, see, e.g., [12,18,22, 25,26], and [34]. More recently, such a notion has been utilized to study the existence of pseudo almost automorphic solutions to various types of differential equations, see for instance [6,12,15,17,18,21,22,25,26,30,32,34], and [40].

In Adamczak [1], the concept of $C^{(n)}$-almost periodicity for real-valued functions was introduced and studied. Next, Bugajewski and N'Guérékata [7] extended such a notion to Banach spaces and then introduced the concept of asymptotic $C^{(n)}$-almost periodicity, which is a natural generalization of the notion of $C^{(n)}$-almost periodicity.

For recent developments on the notion of $C^{(n)}$-almost periodicity and related issues, we refer the reader to for instance [2, 4, 7, 24, 28], and [33] and the references therein. Similarly, in [29] Ezzinbi *et al.*, introduced and studied the notion of $C^{(n)}$-almost automorphy. Recent developments on the concept of $C^{(n)}$-almost automorphy and related issues can be found for instance in [24], and [27].

In a recent paper by Diagana and Nelson [15], the concept of $C^{(n)}$-pseudo almost automorphy (respectively, $C^{(n)}$-pseudo almost periodicity) is introduced. Basic properties of these new functions such as the stability of the convolution or the primitive for $C^{(n)}$-pseudo almost automorphic functions were investigated. Furthermore, Diagana and Nelson [15] obtained the existence of $C^{(n)}$-pseudo almost automorphic solutions to some higher-order systems of differential equations.

The concept of Stepanov-like almost automorphy which is generalization of the classical almost automorphy was introduced in the literature by N'Guérékata and Pankov [36]. Such a notion was then, subsequently, utilized to study the existence of weak Stepanov-like almost automorphic solutions to some parabolic evolution equations. Recently, such a notion gained lots of attention (see, e.g., [19, 23, 30–32]) and has also been generalized (see, e.g., [11, 16], and [18]). In this paper it goes back to introduce and study a new concept called Stepanov-like $C^{(n)}$-pseudo almost automorphy which generalizes the notion of $C^{(n)}$-pseudo almost automorphy (respectively, $C^{(n)}$-pseudo almost periodicity). Basic properties such as the stability of the convolution of those new functions are investigated. It should also be mentioned that the space of $C^{(n)}$-pseudo almost automorphic functions is a subspace of the space of Stepanov-like $C^{(n)}$-pseudo almost automorphic functions (Proposition 2.22).

To illustrate the previous outlined results, we study and obtain the existence of $C^{(m+N)}$-pseudo almost automorphic solutions solutions to the higher-order differential equations

$$w^{(N)}(t) + \sum_{k=0}^{N-1} a_k(t) w^{(k)}(t) = f(t), \quad t \in \mathbb{R}, \tag{1.1}$$

where $a_k : \mathbb{R} \mapsto \mathbb{R}$ for $k = 0, 1, \dots, N-1$ and the function $f : \mathbb{R} \mapsto \mathbb{R}$ is Stepanov-like $C^{(m)}$-pseudo almost automorphic.

2. STEPANOV-LIKE $C^{(n)}$-PSEUDO ALMOST AUTOMORPHIC FUNCTIONS

Let $(\mathbb{X}, \|\cdot\|)$, $(\mathbb{Y}, \|\cdot\|_{\mathbb{Y}})$ be Banach spaces. Let $C(\mathbb{R}, \mathbb{X})$ stand for the collection of continuous functions from $\mathbb{R}$ into $\mathbb{X}$. Similarly, define $C^{(n)}(\mathbb{R}, \mathbb{X})$ as the collection of functions $f : \mathbb{R} \mapsto \mathbb{X}$ such that $f^{(k)}$ exists and belongs to $C(\mathbb{R}, \mathbb{X})$ for $k = 0, 1, 2, \dots, n$. (The symbol $f^{(k)}$ being the k-derivative of f with $f^{(0)}$ corresponding to the continuity of the function f.) Define $BC^{(n)}(\mathbb{R}, \mathbb{X})$ as the collection of all functions $f \in C^{(n)}(\mathbb{R}, \mathbb{X})$ such that

$$\|f\|_{(n)} := \sup_{t \in \mathbb{R}} \sum_{k=0}^{n} \|f^{(k)}(t)\| < \infty.$$

It is not hard to see that $(BC^{(n)}(\mathbb{R}, \mathbb{X}), \|\cdot\|_{(n)})$ is a Banach space.

In this paper, the symbols $f^{(0)}$, $\|\cdot\|_{p,(0)}$, $C^{(0)}(\mathbb{R},\mathbb{X})$, $BS_p^{(0)}(\mathbb{R},\mathbb{X})$, $AS_p^{(0)}(\mathbb{X})$, and $PAAS_p^{(0)}(\mathbb{X})$ stand respectively for f, $\|\cdot\|_{S_p}$, $C(\mathbb{R},\mathbb{X})$, $BS^p(\mathbb{R},\mathbb{X})$, $AS^p(\mathbb{X})$, and $PAAS^p(\mathbb{X})$.

Definition 2.1. The Bochner transform $f^b(t,s)$, $t\in\mathbb{R}$, $s\in[0,1]$, of a function $f:\mathbb{R}\mapsto\mathbb{X}$, is defined by

$$f^b(t,s) := f(t+s).$$

Remark 2.2. Note that a function $\varphi(t,s)$, $t\in\mathbb{R}$, $s\in[0,1]$, is the Bochner transform of a certain function $f(t)$,

$$\varphi(t,s) = f^b(t,s)\,,$$

if and only if $\varphi(t+\tau, s-\tau)=\varphi(s,t)$ for all $t\in\mathbb{R}$, $s\in[0,1]$ and $\tau\in[s-1,s]$.

Definition 2.3 ([36])**.** Let $p\in[1,\infty)$. The space $BS^p(\mathbb{X})$ of all Stepanov bounded functions, with the exponent p, consists of all measurable functions f on $\mathbb{R}$ with values in $\mathbb{X}$ such that $f^b\in L^\infty\big(\mathbb{R}, L^p((0,1);\mathbb{X})\big)$. This is a Banach space with the norm

$$\|f\|_{S^p} = \|f^b\|_{L^\infty(\mathbb{R},L^p)} = \sup_{t\in\mathbb{R}}\left(\int_t^{t+1}\|f(\tau)\|^p\,d\tau\right)^{1/p}.$$

We now introduce the space $BS_p^{(n)}(\mathbb{X})$ as follows.

Definition 2.4. Let $p\in[1,\infty)$ and let $n\in\mathbb{N}$. The space $BS_p^{(n)}(\mathbb{X})$ consists of all functions $f:\mathbb{R}\mapsto\mathbb{X}$ such that $f^{(k)}\in BS^p(\mathbb{X})$ for $k=0,1,\dots,n$. We equip the space $BS_p^{(n)}(\mathbb{X})$ with the norm defined by

$$\|f\|_{p,(n)} := \sup_{t\in\mathbb{R}}\sum_{k=0}^{n}\left(\int_t^{t+1}\|f^{(k)}(\tau)\|^p\,d\tau\right)^{1/p}.$$

Proposition 2.5. *The space $BS_p^{(n)}(\mathbb{X})$ equipped with the norm $\|f\|_{p,(n)}$ is a Banach space.*

Proof. Let $(f_m)_{m\geq0}\subset BS_p^{(n)}(\mathbb{X})$ be a Cauchy sequence. It is clear that $(f_m^{(k)})_{m\geq0}$ is a Cauchy sequence in $BS^p(\mathbb{X})$ for $k=0,1,\dots,n$. Now since $(BS^p(\mathbb{X}),\|\cdot\|_{S^p})$ is a Banach space it follows that there exists a function $g\in BS^p(\mathbb{X})$ that is n-differentiable such that $(f_m^{(k)})_{m\geq0}$ converges to $g^{(k)}$ with respect to the norm $\|\cdot\|_{S^p}$ as $m\to\infty$ for $k=0,1,\dots,n$. Clearly, $\|f_m-g\|_{p,(n)}\to0$ as $m\to\infty$, which yields $BS_p^{(n)}(\mathbb{X})$ equipped with the norm $\|\cdot\|_{p,(n)}$ is a Banach space. $\square$

Definition 2.6. A function $f\in C(\mathbb{R},\mathbb{X})$ is said to be almost automorphic if for every sequence of real numbers $(s'_n)_{n\in\mathbb{N}}$, there exists a subsequence $(s_n)_{n\in\mathbb{N}}$ such that

$$g(t) := \lim_{n\to\infty} f(t+s_n)$$

is well defined for each $t \in \mathbb{R}$, and

$$\lim_{n\to\infty} g(t - s_n) = f(t)$$

for each $t \in \mathbb{R}$.

The collections of almost automorphic functions denoted $AA(\mathbb{X})$, turns out to be a Banach space when equipped with the sup norm $\|\cdot\|_\infty$.

Definition 2.7 ([15, 29])**.** A function $f \in C^{(n)}(\mathbb{R}, \mathbb{X})$ is said to be $C^{(n)}$-almost automorphic if $f^{(k)} \in AA(\mathbb{X})$ for $k = 0, 1, \ldots, n$. The collection of $C^{(n)}$-almost automorphic functions is denoted by $AA^{(n)}(\mathbb{X})$.

Clearly, the following inclusions hold:

$$\ldots \hookrightarrow AA^{(n+2)}(\mathbb{X}) \hookrightarrow AA^{(n+1)}(\mathbb{X}) \hookrightarrow AA^{(n)}(\mathbb{X}) \hookrightarrow \ldots \hookrightarrow AA^{(1)}(\mathbb{X}) \hookrightarrow AA(\mathbb{X}).$$

Definition 2.8 ([15, 29])**.** A jointly continuous function $F : \mathbb{R} \times \mathbb{X} \mapsto \mathbb{Y}$ is said to be $C^{(n)}$-almost automorphic in $t \in \mathbb{R}$ for each $x \in \mathbb{X}$ if $D_t^{(k)} f(t, x) := \frac{\partial^k f}{\partial t^k}(t, x)$ is almost automorphic in $t \in \mathbb{R}$ for each $x \in \mathbb{X}$ where $k = 0, 1, 2, \ldots, n$ with $D_t^{(0)} := \frac{\partial^0 f}{\partial t^0}(t, x) = f(t, x)$. The collection of such functions will be denoted by $AA^{(n)}(\mathbb{X}, \mathbb{Y})$.

Define

$$PAP_0(\mathbb{X}) := \left\{ f \in BC(\mathbb{R}, \mathbb{X}) : \lim_{r\to\infty} \frac{1}{2r} \int_{-r}^{r} \|f(s)\| ds = 0 \right\}.$$

Similarly, $PAP_0(\mathbb{Y}, \mathbb{X})$ will denote the collection of all bounded continuous functions $F : \mathbb{R} \times \mathbb{Y} \mapsto \mathbb{X}$ such that

$$\lim_{r\to\infty} \frac{1}{2r} \int_{-r}^{r} \|F(s, x)\| ds = 0$$

uniformly in $x \in K$, where $K \subset \mathbb{Y}$ is any bounded subset.

Define

$$PAP_0^{(n)}(\mathbb{X}) := \left\{ f \in BC^{(n)}(\mathbb{R}, \mathbb{X}) : \quad f^{(k)} \in PAP_0(\mathbb{X}) \text{ for } k = 0, 1, \ldots, n \right\}$$

and

$$PAP_0^{(n)}(\mathbb{X}, \mathbb{Y}) := \left\{ f \in BC^{(n)}(\mathbb{X}, \mathbb{Y}) : \quad D_t^{(k)} f \in PAP_0(\mathbb{X}, \mathbb{Y}) \text{ for } k = 0, 1, \ldots, n \right\}.$$

Definition 2.9 ([15])**.** A function $f \in BC^{(n)}(\mathbb{R}, \mathbb{X})$ is called $C^{(n)}$-pseudo almost automorphic if it can be expressed as $f = g + \phi$, where $g \in AA^{(n)}(\mathbb{X})$ and $\phi \in PAP_0^{(n)}(\mathbb{X})$. The collection of such functions will be denoted by $PAA^{(n)}(\mathbb{X})$.

Definition 2.10 ([15]). A bounded continuous function $F : \mathbb{R} \times \mathbb{X} \mapsto \mathbb{Y}$ belongs to $PAA^{(n)}(\mathbb{X}, \mathbb{Y})$ whenever it can be expressed as $F = G+\Phi$, where $G \in AA^{(n)}(\mathbb{X}, \mathbb{Y})$ and $\Phi \in PAP_0^{(n)}(\mathbb{X}, \mathbb{Y})$. The collection of such functions will be denoted by $PAA^{(n)}(\mathbb{X}, \mathbb{Y})$.

Theorem 2.11 ([15]). *The space $PAA^{(n)}(\mathbb{X})$ equipped with the norm $\|\cdot\|_{(n)}$ is a Banach space.*

Given two functions $f, g : \mathbb{R} \mapsto \mathbb{R}$, their convolution $f * g$, if it exists, is defined by

$$(f * g)(t) := \int_{-\infty}^{\infty} f(s)g(t-s)ds.$$

Theorem 2.12 ([15]). *If $f \in PAA^{(n)}(\mathbb{R})$ and if $g \in L^1(\mathbb{R})$, then their convolution $f * g \in PAA^{(n)}(\mathbb{R})$.*

Proposition 2.13 ([15]). *If $(f_n)_{n\in\mathbb{N}} \subset PAA(\mathbb{X})$ converges uniformly to f on $\mathbb{R}$, then $f \in PAA(\mathbb{X})$.*

Theorem 2.14 ([15]). *If $f \in PAA^{(n)}(\mathbb{X})$ such that $f^{(n+1)}$ is uniformly continuous, then $f \in PAA^{(n+1)}(\mathbb{X})$.*

Definition 2.15 ([36]). The space $AS^p(\mathbb{X})$ of Stepanov-like almost automorphic functions (or S^p-almost automorphic) consists of all $f \in BS^p(\mathbb{X})$ such that $f^b \in AA\big(L^p((0,1);\mathbb{X})\big)$.

In other words, a function $f \in L^p_{loc}(\mathbb{R};\mathbb{X})$ is said to be S^p-almost automorphic if its Bochner transform $f^b : \mathbb{R} \to L^p((0,1);\mathbb{X})$ is almost automorphic in the sense that for every sequence of real numbers $(s'_n)_{n\in\mathbb{N}}$, there exists a subsequence $(s_n)_{n\in\mathbb{N}}$ and a function $g \in L^p_{loc}(\mathbb{R};\mathbb{X})$ such that

$$\left[\int_t^{t+1} \|f(s_n+s) - g(s)\|^p ds\right]^{1/p} \to 0, \quad \text{and}$$

$$\left[\int_t^{t+1} \|g(s-s_n) - f(s)\|^p ds\right]^{1/p} \to 0$$

as $n \to \infty$ pointwise on $\mathbb{R}$.

We now introduce the following definition.

Definition 2.16. The space $AS_p^{(n)}(\mathbb{X})$ of Stepanov-like $C^{(n)}$-almost automorphic functions (or $S_p^{(n)}$-almost automorphic) consists of all $f \in BS_p^{(n)}(\mathbb{X})$ such that $(f^{(k)})^b \in AA\big(L^p((0,1);\mathbb{X})\big)$ for $k = 0, 1, \ldots, n$.

In other words, a function $f \in BS_p^{(n)}(\mathbb{X})$ is said to be $S_p^{(n)}$-almost automorphic if $(f^{(k)})^b$ is almost automorphic for $k = 0, 1, \ldots, n$ in the sense that for every sequence of real numbers $(s'_m)_{m\in\mathbb{N}}$, there exists a subsequence $(s_m)_{m\in\mathbb{N}}$ and a function $g \in BS_p^{(n)}(\mathbb{X})$ such that

$$\left[\int_t^{t+1} \|f^{(k)}(s_m+s) - g^{(k)}(s)\|^p ds\right]^{1/p} \to 0, \text{ and}$$

$$\left[\int_t^{t+1} \|g^{(k)}(s-s_m) - f^{(k)}(s)\|^p ds\right]^{1/p} \to 0$$

for $k = 0, 1, \ldots, n$, as $m \to \infty$ pointwise on $\mathbb{R}$.

Definition 2.17. A function $F : \mathbb{R}\times\mathbb{X} \mapsto \mathbb{X}, (t, u) \mapsto F(t, u)$ with $F(\cdot, u) \in AS_p^{(n)}(\mathbb{X})$ for each $u \in \mathbb{X}$, is said to be $S_p^{(n)}$-almost automorphic in $t \in \mathbb{R}$ uniformly in $u \in \mathbb{X}$.

The collection of those $S_p^{(n)}$-almost automorphic functions $F : \mathbb{R} \times \mathbb{X} \mapsto \mathbb{X}$ will be denoted by $AS_p^{(n)}(\mathbb{R} \times \mathbb{X}, \mathbb{X})$.

Definition 2.18. The space $PAAS_p^{(n)}(\mathbb{X})$ of Stepanov-like $C^{(n)}$-pseudo almost automorphic functions (or $S_p^{(n)}$-pseudo almost automorphic) consists of all $f \in BS_p^{(n)}(\mathbb{X})$ such that $(f^{(k)})^b \in PAA\big(L^p((0, 1); \mathbb{X})\big)$ for $k = 0, 1, \ldots, n$.

In other words, a function $f \in BS_p^{(n)}(\mathbb{X})$ is said to be $S_p^{(n)}$-pseudo almost automorphic if $f = h + \varphi$ such that for every sequence of real numbers $(s'_m)_{m\in\mathbb{N}}$, there exists a subsequence $(s_m)_{m\in\mathbb{N}}$ and a function $g \in BS_p^{(n)}(\mathbb{X})$ such that

$$\left[\int_t^{t+1} \|h^{(k)}(s_m+s) - g^{(k)}(s)\|^p ds\right]^{1/p} \to 0, \text{ and}$$

$$\left[\int_t^{t+1} \|g^{(k)}(s-s_m) - h^{(k)}(s)\|^p ds\right]^{1/p} \to 0$$

for $k = 0, 1, \ldots, n$, as $m \to \infty$ pointwise on $\mathbb{R}$, and

$$\lim_{r\to\infty} \frac{1}{2r} \int_{-r}^{r} \Big(\int_t^{t+1} \big\|\varphi^{(k)}(\sigma)\big\|^p d\sigma\Big)^{1/p} dt = 0$$

for $k = 0, 1, \ldots, n$.

Definition 2.19. A function $F : \mathbb{R} \times \mathbb{X} \mapsto \mathbb{Y}, (t, u) \mapsto F(t, u)$ with $F(\cdot, u) \in PAAS_p^{(n)}(\mathbb{Y})$ for each $u \in \mathbb{X}$, is said to be $S_p^{(n)}$-pseudo almost automorphic in $t \in \mathbb{R}$ uniformly in $u \in \mathbb{X}$.

The collection of those $S_p^{(n)}$-pseudo almost automorphic functions $F : \mathbb{R} \times \mathbb{X} \mapsto \mathbb{Y}$ will be denoted by $PAAS_p^{(n)}(\mathbb{R} \times \mathbb{X}, \mathbb{Y})$.

The following inclusions hold:

$$\begin{aligned} \ldots \hookrightarrow PAAS_p^{(n+2)}(\mathbb{X}) \hookrightarrow PAAS_p^{(n+1)}(\mathbb{X}) \hookrightarrow PAAS_p^{(n)}(\mathbb{X}) \hookrightarrow \\ \hookrightarrow \ldots \hookrightarrow PAAS_p^{(1)}(\mathbb{X}) \hookrightarrow PAAS_p(\mathbb{X}). \end{aligned}$$

Theorem 2.20. *If $f \in PAAS_p^{(n)}(\mathbb{R})$ and if $g \in L^1(\mathbb{R})$, then their convolution $f * g \in PAAS_p^{(n)}(\mathbb{R})$.*

Proof. Let $f \in PAAS_p^{(n)}(\mathbb{R})$ and let $g \in L^1(\mathbb{R})$. Let $f = h + \varphi$ such that $(h^{(k)})^b = (h^b)^{(k)} \in AA\big(L^p(0, 1; \mathbb{R})\big)$ and $(\varphi^{(k)})^b = (\varphi^b)^{(k)} \in PAP_0\big(L^p((0, 1); \mathbb{R})\big)$ for $k = 0, 1, \ldots, n$. To complete the proof it suffices to show that $[(h * g)^{(k)}]^b \in AA\big(L^p((0, 1); \mathbb{R})\big)$ and $[(\varphi * g)^{(k)}]^b \in PAP_0\big(L^p((0, 1); \mathbb{R})\big)$ for all $k = 0, 1, 2, \ldots, n$. Indeed, using the fact $[(h * g)^{(k)}]^b = [h^{(k)} * g]^b = (h^{(k)})^b * g$ and Theorem 2.12, it follows that $[(h * g)^{(k)}]^b \in AA\big(L^p((0, 1); \mathbb{R})\big)$ for all $k = 0, 1, 2, \ldots, n$. Similarly, from $[(\varphi * g)^{(k)}]^b = [\varphi^{(k)} * g]^b = (\varphi^{(k)})^b * g$ and Theorem 2.12 it follows that $[(\varphi * g)^{(k)}]^b \in PAP_0\big(L^p((0, 1); \mathbb{R})\big)$ for all $k = 0, 1, 2, \ldots, n$. This completes the proof. □

Proposition 2.21. *The space $PAAS_p^{(n)}(\mathbb{X})$ equipped with the norm $\|\cdot\|_{p,(n)}$ is a Banach space.*

Proof. The proof is based upon the fact $PAAS_p^{(n)}(\mathbb{X})$ is a closed subspace of $BS_p^{(n)}(\mathbb{X})$. □

Proposition 2.22. *If $f \in PAA^{(n)}(\mathbb{X})$, then $f \in PAAS_p^{(n)}(\mathbb{X})$. That is, $PAA^{(n)}(\mathbb{X}) \subset PAAS_p^{(n)}(\mathbb{X})$.*

Proof. Let $f = h + \varphi$ where $h \in AA^{(n)}(\mathbb{X})$ and $\varphi \in PAP_0^{(n)}(\mathbb{X})$. Clearly, $f^{(k)} \in BS^p(\mathbb{X})$ for $k = 0, 1, \ldots, n$ and hence $f \in BS_p^{(n)}(\mathbb{X})$. To complete the proof it suffices to show that $(h^{(k)})^b \in AA(L^p((0, 1); \mathbb{X}))$ and $(\varphi^{(k)})^b \in PAP_0(L^p((0, 1); \mathbb{X}))$ for $k = 0, 1, 2, \ldots, n$. Using the fact that $AA(\mathbb{X}) \subset AS^p(\mathbb{X})$ for $p \in [1, \infty)$ (see [36]) it follows that $(h^{(k)})^b \in AA(L^p((0, 1); \mathbb{X}))$ for $k = 0, 1, 2, \ldots, n$. Similarly, Diagana [16, Proposition 2.12] has shown that if $k \in PAP_0(\mathbb{X})$, then $k^b \in PAP_0(L^p((0, 1); \mathbb{X}))$. Using that it readily follows that $(\varphi^{(k)})^b \in PAP_0(L^p((0, 1); \mathbb{X}))$ for $k = 0, 1, 2, \ldots, n$. □

Example 2.23. Let $p = 1$ and let $\varepsilon \in AA^{(3)}(\mathbb{R})$ such that $\inf_{t \in \mathbb{R}} \varepsilon'''(t) = \delta_0 > 0$. We give an example of a function $f \in PAAS_1^{(2)}(\mathbb{R})$ such that $f \notin PAAS_1^{(3)}(\mathbb{R})$. Indeed, consider the function f defined by

$$f(t) = \sum_{k=1}^{\infty} \frac{\varepsilon(kt)}{k^4} + \frac{1}{1 + t^2}.$$

Setting

$$\varphi(t) = \sum_{k=1}^{\infty} \frac{\varepsilon(kt)}{k^4} \quad \text{and} \quad h(t) = \frac{1}{1 + t^2},$$

we claim that:

(i) $\varphi \in AS_1^{(2)}(\mathbb{R})$ while $\varphi \notin AS_1^{(3)}(\mathbb{R})$; and
(ii) $h \in PAP_0^{(2)}(\mathbb{R})$.

Indeed,

$$\varphi'(t) = \sum_{k=1}^{\infty} \frac{\varepsilon'(kt)}{k^3} \quad \text{and} \quad \varphi''(t) = \sum_{k=1}^{\infty} \frac{\varepsilon''(kt)}{k^2}$$

for all $t \in \mathbb{R}$.

Clearly, $\varphi \in BS_1^{(2)}(\mathbb{R})$ as it can be easily shown that $\varphi, \varphi', \varphi'' \in BS^1(\mathbb{R})$. Moreover, $\varphi, \varphi', \varphi'' \in AA(\mathbb{R})$. Consequently, $\varphi \in AS_1^{(2)}(\mathbb{R})$.

Now

$$\varphi'''(t) = \sum_{k=1}^{\infty} \frac{\varepsilon'''(kt)}{k}.$$

Clearly, $\varphi''' \notin BS^1(\mathbb{R})$. Therefore, $\varphi \in AS_1^{(2)}(\mathbb{R})$ while $\varphi \notin AS_1^{(3)}(\mathbb{R})$.

The fact $h \in PAP_0^{(2)}(\mathbb{R})$ has been shown in Diagana and Nelson [15]. However for the sake of clarity, we reproduce it here. Indeed, it is readily seen that h belongs to $PAP_0(\mathbb{R})$. Now the functions

$$h'(t) = -\frac{2t}{(1+t^2)^2}, \quad h''(t) = \frac{-2-10t^2}{(1+t^2)^3}$$

are bounded continuous functions with $|h'(t)| \leq 1$ and $|h''(t)| \leq 12$ for all $t \in \mathbb{R}$. Further, it is easily seen that both h' and h'' belongs to $PAP_0(\mathbb{R})$ and hence $h \in PAP_0^{(2)}(\mathbb{R})$.

In view of the above it follows that $f \in PAAS_1^{(2)}(\mathbb{R})$ while $f \notin PAAS_1^{(3)}(\mathbb{R})$.

3. EXISTENCE OF $C^{(m)}$-PSEUDO ALMOST AUTOMORPHIC SOLUTIONS

In this section we study the existence of $C^{(m)}$-pseudo almost automorphic solutions to the nonautonomous higher-order differential equations (1.1). For that, we rewrite it as a first-order differential equation in $\mathbb{R}^N$ involving an $N \times N$ square matrix $\mathcal{A}(t)$. Indeed, if u is N-times differentiable and setting

$$z := \begin{pmatrix} u \\ u' \\ u'' \\ \vdots \\ u^{(N-1)} \end{pmatrix} \in \mathbb{R}^N,$$

then (1.1) can be rewritten in the following form:

$$z'(t) = \mathcal{A}(t)z(t) + F(t), \ t \in \mathbb{R}, \tag{3.1}$$

where $\mathcal{A}(t)$ is the $N \times N$ square matrix given by

$$\mathcal{A}(t) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \\ 0 & 0 & 1 & \cdots\cdot & 0 \\ \\ \vdots & & \ddots & \ddots & \\ \\ -a_0(t) & -a_1(t) & \cdots & \cdots & -a_{N-1}(t) \end{pmatrix} \tag{3.2}$$

and the function F appearing in (1.1) is defined on $\mathbb{R}$ by

$$F(t) := \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ f(t) \end{pmatrix}.$$

Let $\{A(t)\}_{t\in\mathbb{R}}$ be an $N \times N$ square matrix and consider the differential equation

$$z'(t) = A(t)z(t) + g(t), \quad t \in \mathbb{R}, \tag{3.3}$$

and its corresponding homogeneous equation

$$z'(t) = A(t)z(t), \quad t \in \mathbb{R}, \tag{3.4}$$

where $g : \mathbb{R} \mapsto \mathbb{R}^N$ is continuous.

Definition 3.1. The homogeneous equation (3.4) is said to be to have an exponential dichotomy if there exist a projection P and the constants $K, \delta > 0$ such that:

(i) $\|X(t)PX^{-1}(s)\| \leq Ke^{-\delta(t-s)}$ for all $t, s \in \mathbb{R}$ and $t \geq s$; and
(ii) $\|X(t)QX^{-1}(s)\| \leq Ke^{-\delta(s-t)}$ for all $t, s \in \mathbb{R}$ and $t \leq s$,

where $Q = I - P$ and $X(t)$ is a fundamental solution to (3.4) satisfying $X(0) = I$ (I being the identity matrix for $\mathbb{R}^N$).

If (3.4) has an exponential dichotomy, we then define

$$\Gamma(t,s) = \begin{cases} X(t)PX^{-1}(s) & \text{if } t \geq s, \\ \\ -X(t)QX^{-1}(s) & \text{if } s \geq t. \end{cases}$$

Clearly,

$$\|\Gamma(t,s)\| \leq \begin{cases} Ke^{-\delta(t-s)} & \text{if } t \geq s, \\ \\ Ke^{-\delta(s-t)} & \text{if } s \geq t. \end{cases}$$

We require the following assumptions:

(H1) g is Stepanov-like $C^{(m)}$-pseudo almost automorphic; and
(H2) $\Gamma(t,s)u \in bAA(\mathbb{R},\mathbb{R}^N)$ uniformly for all u in any bounded subset of $\mathbb{R}^N$.

Theorem 3.2. *If* (3.4) *has exponential dichotomy and if assumptions* (H1)–(H2) *hold, then* (3.3) *has a unique $C^{(m)}$-pseudo almost automorphic solution.*

Proof. Let $X(t)$ be a fundamental solution to (3.4) satisfying $X(0) = I$ and suppose there exists a projection P and the constants $K, \delta > 0$ such that

$$\|X(t)PX^{-1}(s)\| \leq Ke^{-\delta(t-s)} \tag{3.5}$$

for all $t, s \in \mathbb{R}$ and $t \geq s$, and

$$\|X(t)QX^{-1}(s)\| \leq Ke^{-\delta(s-t)} \tag{3.6}$$

for all $t, s \in \mathbb{R}$ and $t \leq s$, where $Q = I - P$.

According to Coppel [13], the only bounded solution to (3.3) is given by

$$z(t) = \int_{-\infty}^{\infty} \Gamma(t,s)g(s)ds.$$

Let $q > 1$ such that $p^{-1} + q^{-1} = 1$. Consider for each $n \in \mathbb{N}$, the integral

$$u_n(t) := \int_{t-n}^{t-n+1} \Gamma(t,s)g(s)ds = \int_{n-1}^{n} \Gamma(t,t-s)g(t-s)ds.$$

Now

$$\|u_n\| \leq K \int_{t-n}^{t-n+1} e^{-\omega(t-s)}\|g(s)\|ds.$$

Using the Hölder's inequality we obtain

$$\begin{aligned}
\|u_n\| &\leq K\Big(\int_{t-n}^{t-n+1} e^{-q\omega(t-s)}ds\Big)^{1/q}\Big(\int_{t-n}^{t-n+1} \|g(s)\|^p ds\Big)^{1/p} = \\
&= K\Big[\frac{e^{-nq\omega}}{q\omega}\Big(e^{q\omega} - 1\Big)\Big]^{1/q}\|g\|_{S^p} \leq \\
&\leq Ke^{-n\omega}\Big[\frac{e^{q\omega} + 1}{q\omega}\Big]^{1/q}\|g\|_{S^p} = \\
&= c(K,\omega,q)e^{-n\omega}\|g\|_{S^p}.
\end{aligned}$$

Since the series $c(K,\omega,q)\sum_{n=1}^{\infty} e^{-n\omega}$ is convergent, it follows from the Weierstrass test that

$$\sum_{n=1}^{\infty} u_n(t)$$

is uniformly convergent on $\mathbb{R}$ and set $u(t) = \sum_{n=1}^{\infty} u_n(t)$.

Note that

$$u(t) = \int_{-\infty}^{\infty} \Gamma(t,s)g(s)ds.$$

Moreover, $u \in C(\mathbb{R},\mathbb{R}^N)$ and for any $t \in \mathbb{R}$,

$$\|u(t)\| \leq \sum_{n=1}^{\infty} \|u_n(t)\| \leq c'(K,\omega,q)\|g\|_{S^p}$$

where $c'(K,\omega,q)$ is a constant.

Let us show that u_n is $C^{(m)}$-pseudo almost automorphic. Since g is Stepanov-like $C^{(m)}$-pseudo almost automorphic, let $g = g_1 + g_2$ where $(g_1^{(r)})^b \in AA(L^p((0,1),\mathbb{R}^N))$ and $(g_2^{(r)})^b \in PAP_0(L^p((0,1),\mathbb{R}^N))$ for $r = 0,1,2,\dots,m$. Using the fact $(g_1^{(r)})^b \in AA(L^p((0,1),\mathbb{R}^N))$ for $r = 0,1,2,\dots,m$ it follows that for every sequence of real numbers $(\tau_k')_{k\in\mathbb{N}}$ there exists a subsequence $(\tau_k)_{k\in\mathbb{N}}$ and an m-differentiable function h_1 such that

$$\lim_{k\to\infty}\int_0^1 \|g_1^{(r)}(t+\tau_k+s)-h_1^{(r)}(t+s)\|^p ds = \lim_{k\to\infty}\int_0^1 \|h_1^{(r)}(t-\tau_k+s)-g_1^{(r)}(t+s)ds\|^p = 0$$

for each $t \in \mathbb{R}$ and for $r = 0,1,2,\dots,m$.

Similarly, using (H2), we can assume that

$$\Lambda(t,s)h = \lim_{k\to\infty} \Gamma(t+\tau_k, s+\tau_k)h$$

is well-defined for each $h \in \mathbb{R}^N$ and $t,s \in \mathbb{R}$, and

$$\Gamma(t,s)h = \lim_{k\to\infty} \Lambda(t-\tau_k, s-\tau_k)h$$

is well-defined for each $h \in \mathbb{R}^N$ and $t,s \in \mathbb{R}$.

Setting

$$I_k^p(t) := \Big(\int_{n-1}^{n} \|\Gamma(t+\tau_k, t+\tau_k-s)(g_1^{(r)}(t+\tau_k-s) - h_1^{(r)}(t-s))\|^p ds\Big)^{1/p}$$

and

$$J_k^p(t) := \Big(\int_{n-1}^{n} \|(\Gamma(t+\tau_k, t+\tau_k - s) - \Lambda(t, t-s))h_1^{(r)}(t-s)\|^p ds\Big)^{1/p}$$

for $r = 0, 1, 2, \ldots, m$, one can see that

$$\Big(\int_{n-1}^{n} \|\Gamma(t+\tau_k, t+\tau_k - s)g_1^{(r)}(t+\tau_k - s) - \Lambda(t, t-s)h_1^{(r)}(t-s)\|^p ds\Big)^{1/p} \leq$$
$$\leq I_k^p(t) + J_k^p(t)$$

for $r = 0, 1, 2, \ldots, m$.

Using (3.5) and the Lebesgue Dominated Convergence theorem, one can easily see that

$$I_k^p(t) \to 0 \text{ as } k \to \infty, \ t \in \mathbb{R}.$$

Similarly, using (H2) it follows that

$$J_k^p(t) \to 0 \text{ as } k \to \infty, \ t \in \mathbb{R}.$$

Consequently,

$$\lim_{k\to\infty} \int_{n-1}^{n} \|\Gamma(t+\tau_k, t+\tau_k - s)g_1^{(r)}(t+\tau_k - s) - \Lambda(t, t-s)h_1^{(r)}(t-s)\|^p ds = 0$$

for $r = 0, 1, 2, \ldots, m$.

Using similar ideas as the previous ones, one can show that

$$\lim_{k\to\infty} \int_{n-1}^{n} \|\Lambda(t-\tau_k, t-\tau_k - s)h_1^{(r)}(t-\tau_k - s) - \Gamma(t, t-s)g_1^{(r)}(t-s)\|^p ds = 0$$

for $r = 0, 1, 2, \ldots, m$.

Set

$$v_n(t) := \int_{t-n}^{t-n+1} \Gamma(t,s)g_2^{(r)}(s)ds = \int_{n-1}^{n} \Gamma(t, t-s)g_2^{(r)}(t-s)ds$$

for $r = 0, 1, 2, \ldots, m$.

Now

$$\|v_n\| \leq K \int_{t-n}^{t-n+1} e^{-\omega(t-s)}\|g_2^{(r)}(s)\|ds$$

for $r = 0, 1, 2, \dots, m$. Using the Hölder's inequality we obtain

$$\|v_n\| \le K\Big(\int\limits_{t-n}^{t-n+1} e^{-q\omega(t-s)}ds\Big)^{1/q}\Big(\int\limits_{t-n}^{t-n+1} \|g_2^{(r)}(s)\|^p ds\Big)^{1/p} =$$

$$= K\Big[\frac{e^{-nq\omega}}{q\omega}\Big(e^{q\omega}-1\Big)\Big]^{1/q}\Big(\int\limits_{t-n}^{t-n+1} \|g_2^{(r)}(s)\|^p ds\Big)^{1/p} \le$$

$$\le Ke^{-n\omega}\Big[\frac{e^{q\omega}+1}{q\omega}\Big]^{1/q}\Big(\int\limits_{t-n}^{t-n+1} \|g_2^{(r)}(s)\|^p ds\Big)^{1/p} =$$

$$= c(K,\omega,q)e^{-n\omega}\Big(\int\limits_{t-n}^{t-n+1} \|g_2^{(r)}(s)\|^p ds\Big)^{1/p}$$

and hence $v_n \in PAP_0(\mathbb{R}^n)$ as $(g_2^{(r)})^b \in PAP_0(L^p((0,1),\mathbb{R}^N))$ for $r = 0, 1, 2, \dots, m$. Furthermore, since the series $c(K,\omega,q)\sum_{n=1}^{\infty} e^{-n\omega}$ is convergent, it follows from the Weierstrass test that

$$\sum_{n=1}^{\infty} v_n(t)$$

is uniformly convergent on $\mathbb{R}$ and set $v(t) = \sum\limits_{n=1}^{\infty} v_n(t)$.

Now

$$v(t) = \int\limits_{-\infty}^{\infty} \Gamma(t,s) g_2^{(r)}(s)ds$$

for $r = 0, 1, 2, \dots, m$. Moreover, $v \in C(\mathbb{R},\mathbb{R}^N)$ and for any $t \in \mathbb{R}$,

$$\|v(t)\| \le \sum_{n=1}^{\infty} \|v_n(t)\| \le c'(K,\omega,q)\|g_2^{(r)}\|_{S^p}$$

for $r = 0, 1, 2, \dots, m$, where $c'(K,\omega,q)$ is a constant. Thus the uniform limit $v(t) = \sum_{n=1}^{\infty} v_n(t) \in PAP_0(\mathbb{R}^N)$. The proof is complete. $\square$

We suppose that

(H3) $t \to f(t)$ is Stepanov-like $C^{(m)}$-pseudo almost automorphic.

Theorem 3.3. *If the homogeneous equation associated with* (3.1) *has exponential dichotomy and if assumptions* (H2)–(H3) *hold, then* (1.1) *has a unique $C^{(m+N)}$-pseudo almost automorphic solution.*

Proof. Using Theorem 3.2 it follows that (3.1) has a unique $C^{(m)}$-pseudo almost automorphic solution given by

$$t \to z(t) := \begin{pmatrix} w(t) \\ w'(t) \\ w''(t) \\ \vdots \\ w^{(N-1)}(t) \end{pmatrix}.$$

Therefore, $t \to w(t), t \to w'(t), \dots, t \to w^{(N)}(t) \in PAA(\mathbb{R})$ and hence $t \to w(t) \in PAA^{(N+m)}(\mathbb{R})$, i.e., (1.1) has a unique $C^{(N+m)}$-pseudo almost automorphic solution w. □

REFERENCES

[1] M. Adamczak, *C^n-almost periodic functions*, Comment. Math. (Prace Mat.) **37** (1997), 1–12.

[2] M. Adamczak, S. Stoinski, *On the $(NC^{(n)})$-almost periodic functions*, Proc. of the 6th Conference on Functions Spaces (R. Grząślewicz, Cz. Ryll-Nardzewski, H. Hudzik, and J. Musielak, eds), World Scientific Publishing, New Jersey, 2003, 39–48.

[3] R.P. Agarwal, B. de Andrade, C. Cuevas, *Weighted pseudo-almost periodic solutions of a class of semilinear fractional differential equations*, Nonlinear Anal. (RWA) **11** (2010) 5, 3532–3554.

[4] J.B. Baillon, J. Blot, G.M. N'Guérékata, D. Pennequin, *On $C^{(n)}$-almost periodic solutions to some nonautonomous differential equations in Banach spaces*, Annales Societatis Mathematicae Polonae, Serie 1, XLVI (2006) 2, 263–273.

[5] N. Boukli-Hacenea, K. Ezzinbi, *Weighted pseudo almost periodic solutions for some partial functional differential equations*, Nonlinear Anal. **71** (2009) 9, 3612–3621.

[6] N. Boukli-Hacenea, K. Ezzinbi, *Weighted pseudo-almost automorphic solutions for some partial functional differential equations*, Nonlinear Anal. (RWA) **12** (2011) 1, 562–570.

[7] D. Bugajewski, G.M. N'Guérékata, *On some classes of almost periodic functions in abstract spaces*, Int. J. Math. Math. Sci. **61** (2004), 3237–3247.

[8] H. Bohr, *Almost Periodic Functions*, Chelsea Publishing Company, New York, 1947.

[9] D. Bugajewski, T. Diagana, *Almost automorphy of the convolution operator and applications to differential and functional differential equations*, Nonlinear Stud. **13** (2006) 2, 129–140.

[10] D. Bugajewski, T. Diagana, C.M. Mahop, *Asymptotic and pseudo almost periodicity of the convolution operator and applications to differential and integral equations*, Z. Anal. Anwend. **25** (2006) 3, 327–340.

[11] Y.K. Chang, Z.H. Zhao, G.M. N'Guérékata, R. Ma, *Stepanov-like almost automorphy for stochastic processes and applications to stochastic differential equations*, Nonlinear Anal. **12** (2011) 2, 1130–1139.

[12] P. Cieutat, K. Ezzinbi, *Existence, uniqueness and attractiveness of a pseudo almost automorphic solutions for some dissipative differential equations in Banach spaces*, J. Math. Anal. Appl. **354** (2009) 2, 494–506.

[13] W.A. Coppel, *Dichotomies in stability theory*, Lecture Notes in Mathematics, vol. 629, New York, 1978.

[14] Ju.L. Daleckii, M.G. Krein, *Stability of solutions of differential equations in Banach space*, translated from the Russian by S. Smith, translations of Mathematical Monographs, vol. 43, American Mathematical Society, Providence, R.I., 1974.

[15] T. Diagana, V. Neslon, *$C^{(n)}$-pseudo almost automorphy and its applications to some higher-order differential equations*, Nonlinear Stud. (in press).

[16] T. Diagana, *Stepanov-like pseudo-almost periodicity and its applications to some nonautonomous differential equations*, Nonlinear Anal. **69** (2008), 4277–4285.

[17] T. Diagana, *Pseudo-almost automorphic solutions to some classes of nonautonomous partial evolution equations*, Differ Equ. Appl. **1** (2009) 4, 561–582.

[18] T. Diagana, *Existence of pseudo-almost automorphic solutions to some abstract differential equations with S^p-pseudo-almost automorphic coefficients*, Nonlinear Anal. **70** (2009) 11, 3781–3790.

[19] T. Diagana, G.M. Mophou, G.M. N'Guérékata, *Existence of weighted pseudo-almost periodic solutions to some classes of differential equations with S^p-weighted pseudo-almost periodic coefficients*, Nonlinear Anal. **72** (2010), 430–438.

[20] T. Diagana, *Weighted pseudo-almost periodic functions and applications*, C.R. Math. Acad. Sci. Paris, **343** (2006) 10, 643–646.

[21] T. Diagana, *Existence of weighted pseudo almost periodic solutions to some classes of hyperbolic evolution equations*, J. Math. Anal. Appl. **350** (2009) 1, 18–28.

[22] H.S. Ding, J. Liang, G.M. N'Guerekata, T.J. Xiao, *Pseudo-almost periodicity of some nonautonomous evolution equations with delay*, Nonlinear Anal. **67** (2007), 1412–1418.

[23] H.S. Ding, J. Liang, T.J. Xiao, *Some properties of Stepanov-like almost automorphic functions and applications to abstract evolution equations*, Appl. Anal. **88** (2009) 7, 1079–1091.

[24] A. Elazzouzi, *$C^{(n)}$-almost periodic and $C^{(n)}$-almost automorphic solutions for a class of partial functional differential equations with finite delay*, Nonlinear Anal. (Hybrid Systems) **4** (2010) 4, 672–688.

[25] K. Ezzinbi, S. Fatajou, G.M. N'Guérékata, *Pseudo almost automorphic solutions to some neutral partial functional differential equations in Banach space*, Nonlinear Anal. **70** (2009) 4, 1641–1647.

[26] K. Ezzinbi, S. Fatajou, G.M. N'Guérékata, *Pseudo almost automorphic solutions for dissipative differential equations in Banach spaces*, J. Math. Anal. Appl. **351** (2009) 2, 765–772.

[27] K. Ezzinbi, V. Nelson, G.M. N'Guérékata, *$C^{(n)}$-almost automorphic solutions of some nonautonomous differential equations*, Cubo **10** (2008) 2, 61–74.

[28] K. Ezzinbi, S. Fatajou, G.M. N'Guérékata, *Massera type theorem for the existence of $C^{(n)}$-almost periodic solutions for partial functional differential equations with infinite delay*, Nonlinear Anal. **69** (2008) 4, 1413–1424.

[29] K. Ezzinbi, S. Fatajou, G.M. N'Guérékata, *$C^{(n)}$-almost automorphic solutions for partial neutral functional differential equations*, Appl. Anal. **86** (2007) 9, 1127–1146.

[30] Z. Fan, J. Liang, T.J. Xio, *On Stepanov-like (pseudo) almost automorphic functions*, Nonlinear Anal. **74** (2011) 8, 2853–2861.

[31] Z. Hu, Z. Jin, *Stepanov-like pseudo-almost periodic mild solutions to perturbed nonautonomous evolution equations with infinite delay*, Nonlinear Anal. **71** (2009) 11, 5381–5391.

[32] Z. Hu, Z. Jin, *Stepanov-like pseudo almost automorphic mild solutions to nonautonomous evolution equations*, Nonlinear Anal. **71** (2009) 5–6, 2349–2360.

[33] J. Liang, L. Maniar, G.M. N'Guérékata, T.J. Xiao, *Existence and uniqueness of $C^{(n)}$-almost periodic solutions to some ordinary differential equations*, Nonlinear Anal. **66** (2007) 9, 1899–1910.

[34] J. Liang, G.M. N'Guérékata, T-J. Xiao, J. Zhang, *Some properties of pseudo almost automorphic functions and applications to abstract differential equations*, Nonlinear Anal. **70** (2009) 7, 2731–2735.

[35] J.H. Liu, G.M. N'Guérékata, N.V. Minh, *Topics on stability and periodicity in abstract differential equations*, Series on Concrete and Applicable Mathematics, vol. 6, World Scientific, 2008.

[36] G.M. N'Guérékata, A. Pankov, *Stepanov-like almost automorphic functions and monotone evolution equations*, Nonlinear Anal. **68** (2008) 9, 2658–2667.

[37] G.M. N'Guérékata, A. Pankov, *Integral operators in spaces of bounded almost periodic and almost automorphic functions*, Differential and Integral Equations **21** (2008) 11–12, 1155–1176.

[38] G.M. N'Guérékata, *Almost Automorphic Functions and Almost Periodic Functions in Abstract Spaces*, Kluwer Academic / Plenum Publishers, New York-London-Moscow, 2001.

[39] G.M. N'Guérékata, *Topics in Almost Automorphy*, Springer, New York, Boston, Dordrecht, Lodon, Moscow, 2005.

[40] T-J. Xiao, J. Liang, J. Zhang, *Pseudo-almost automorphic solutions to semilinear differential equations in Banach spaces*, Semigroup Forum **76** (2008) 3, 518–524.

[41] T.J. Xiao, X-X. Zhu, J. Liang, *Pseudo-almost automorphic mild solutions to nonautonomous differential equations and applications*, Nonlinear Anal. **70** (2009) 11, 4079–4085.

[42] C.Y. Zhang, *Integration of vector-valued pseudo almost periodic functions*, Proc. Amer. Math. Soc. **121** (1994), 167–174.

Toka Diagana
tdiagana@howard.edu

Howard University
Department of Mathematics
2441 6th Street N.W.
Washington, D.C. 20059, USA

Valerie Nelson
vnelson@howard.edu

Howard University
Department of Mathematics
2441 6th Street N.W.
Washington DC 20059, USA

Gaston M. N'Guérékata
Gaston.N'Guerekata@morgan.edu

Morgan State University
Department of Mathematics
Baltimore, MD 21251, USA

Received: August 20, 2011.
Revised: November 28, 2011.
Accepted: November 28, 2011.