ON THE EXISTENCE OF THREE SOLUTIONS FOR QUASILINEAR ELLIPTIC PROBLEM

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Abstract. We consider a quasilinear elliptic problem of the type $-\Delta_p u = \lambda(f(u)+\mu g(u))$ in Ω, $u|_{\partial\Omega} = 0$, where $\Omega \subset \mathbb{R}^N$ is an open and bounded set, f, g are continuous real functions on $\mathbb{R}$ and λ, $\mu \in \mathbb{R}$. We prove the existence of at least three solutions for this problem using the so called three critical points theorem due to Ricceri.

Keywords: critical point, elliptic problem, minimax inequality, p-Laplacian, three critical points theorem, weak solution.

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1. INTRODUCTION

In this paper we prove that the problem

$$\begin{cases} -\Delta_p u = \lambda(f(u) + \mu g(u)) \text{ in } \Omega, \\ u|_{\partial\Omega} = 0, \end{cases} \tag{1.1}$$

has at least three weak solutions in the Sobolev space $W_0^{1,p}(\Omega)$. Here Δ_p stands for the p-Laplacian defined by $\Delta_p := \operatorname{div}\big(|\nabla \cdot|^{p-2}\nabla \cdot\big)$ with $p \in (1, +\infty)$.

The main result of the paper is the following.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^N$ be an open and bounded set with smooth boundary and let f, $g\colon \mathbb{R} \to \mathbb{R}$ be continuous functions. Put*

$$F(\xi) := \int_0^{\xi} f(t)\,dt, \qquad G(\xi) := \int_0^{\xi} g(t)\,dt.$$

Assume that $\sup_{\xi\in\mathbb{R}} F(\xi) > 0$ *and that there exists four positive constants* a, q, s, γ *with* $s \in [1,p)$, $\gamma \in \big(p,p^*\big)$ *and* $q \in \big(0,p^*-1\big)$, *where*

$$p^* := \begin{cases} \frac{Np}{N-p} & \text{if } p < N, \\ +\infty & \text{if } p \geq N, \end{cases}$$

such that

$$\max\big\{|f(\xi)|,|g(\xi)|\big\} \leq \ a\big(1+|\xi|^q\big), \quad \forall \xi \in \mathbb{R}, \tag{1.2}$$

$$\max\{F(\xi),|G(\xi)|\} \leq \ a\big(1+|\xi|^s\big), \quad \forall \xi \in \mathbb{R}, \tag{1.3}$$

and

$$\limsup_{\xi\to 0} \frac{F(\xi)}{|\xi|^\gamma} < +\infty. \tag{1.4}$$

Then, there exists $\delta > 0$ *such that, for each* $\mu \in [-\delta,\delta]$, *there exist* $\rho > 0$ *and an open nonempty interval* $\Lambda \subset [0,+\infty)$ *such that, for each* $\lambda \in \Lambda$, *problem* (1.1) *has at least three weak solutions in* $W_0^{1,p}(\Omega)$ *whose norms are less than* ρ.

Existence and multiplicity results for problems involving the p-Laplacian or $p(x)$-Laplacian have been investigated in recent years by many mathematicians and three critical points theorem has been often used as a main tool in their proofs. Problem (1.1) has been studied by Ricceri in [15,16] when $p = 2$ while in [13,14] he formulated modified versions of the three critical points theorem and the existence theorems for both Dirichlet and Neumann problems with the p-Laplacian. Bonanno and Livrea obtained in [3] the existence of at least three weak solutions for the Dirichlet problem with the p-Laplacian when $p > N$ and Carathéodory function satisfying appropriate growth conditions. By the similar assumptions Bonanno and Candito established in [1] the same result for the Neumann problem with the p-Laplacian. Existence of three solutions for the Dirichlet problem involving the p-Laplacian was also proved by Bonanno and Giovanelli in [2] by using the non-smooth version of the three critical points theorem established by Marano-Motreanu in [10]. On the other hand there are similar results for the problems driven by the $p(x)$-Laplace operator, for example Liu obtained in [9] the existence of at least three solutions for both Dirichlet and Neumann problems in which functions occurred satisfy the subcritical growth condition. In [11] Mihăilescu proved the existence of at least three weak solutions for Neumann problem under the assumptions that $\inf_{x\in\overline{\Omega}} p(x) > N \geq 3$ and the right hand side nonlinearity has the form $f(x,t) = |t|^{q(x)-2}t - t$, where $q \in \big\{h \in C\big(\overline{\Omega}\big) : h(x) > 1 \quad \forall x \in \overline{\Omega}\big\}$ and $2 < q(x) < \inf_{x\in\overline{\Omega}} p(x)$ for any $x \in \overline{\Omega}$. A few years later Wang, Fan and Ge established in [18] analogous result as above for more general f. Another multiplicity result for anisotropic variable exponent problems can be found in Stancu-Dumitru [17]. Finally, let us mention the papers of Gasiński and Papageorgiou [4–7] where the multiplicity results (existence of at least three or five nontrivial solutions) for Dirichlet quasilinear boundary value problems are obtained using different methods than presented here (critical point theory based on the minimax theorems due to Chang).

This paper generalizes the result of Ricceri [16, Theorem 4] to a general p-Laplacian with $p > 1$. Here we do not require the relation between N and p but we assume that the functions which occur in the right hand side of the studied equation satisfy subcritical growth conditions together with their antiderivatives.

2. PRELIMINARIES

In this section we recall some notions and facts needed in the proof of the main theorem.

Theorem 2.1 ([16, Theorem 1]). *Let X be a separable and reflexive real Banach space, $\Phi\colon X \to \mathbb{R}$ continuously Gâteaux differentiable and a sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on X^*, $\Psi\colon X \to \mathbb{R}$ continuously Gâteaux differentiable functional whose Gâteaux derivative is compact and $I \subset \mathbb{R}$ an interval. Assume that*

$$\lim_{\|x\|\to+\infty} (\Phi(x) + \lambda\Psi(x)) = +\infty \quad \textit{for all} \quad \lambda \in I,$$

and that there exists a continuous concave function $h\colon I \to \mathbb{R}$ such that

$$\sup_{\lambda\in I} \inf_{x\in X} (\Phi(x) + \lambda\Psi(x) + h(\lambda)) < \inf_{x\in X} \sup_{\lambda\in I} (\Phi(x) + \lambda\Psi(x) + h(\lambda)).$$

Then there exist $\rho > 0$ and an open interval $\Lambda \subset I$ such that, for each $\lambda \in \Lambda$, the equation

$$\Phi'(x) + \lambda\Psi'(x) = 0$$

has at least three solutions in X whose norms are less than ρ.

Proposition 2.2 ([15, Proposition 3.1]). *Let X be a nonempty set, and Φ, Ψ two real functions on X. Assume that there are $r > 0$ and x_0, $x_1 \in X$ such that*

$$\Phi(x_0) = \Psi(x_0) = 0, \qquad \Phi(x_1) > r, \qquad \sup_{x\in\Phi^{-1}((-\infty,r])} \Psi(x) < r\frac{\Psi(x_1)}{\Phi(x_1)}.$$

Then for each ρ satisfying

$$\sup_{x\in\Phi^{-1}((-\infty,r])} \Psi(x) < \rho < r\frac{\Psi(x_1)}{\Phi(x_1)},$$

one has

$$\sup_{\lambda\geq 0} \inf_{x\in X} (\Phi(x) + \lambda(\rho - \Psi(x))) < \inf_{x\in X} \sup_{\lambda\geq 0} (\Phi(x) + \lambda(\rho - \Psi(x))).$$

Proposition 2.3 ([12, Proposition B.1]). *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and let z satisfy:*

(i) $z \in C\big(\overline{\Omega} \times \mathbb{R}; \mathbb{R}\big)$,
(ii) *there are constants b_1, $b_2 \geq 1$ and a_1, $a_2 \geq 0$ such that*

$$|z(x,\xi)| \leq \ a_1 + a_2|\xi|^{\frac{b_1}{b_2}}, \quad \forall (x,\xi) \in \overline{\Omega} \times \mathbb{R}.$$

Then the map $v \mapsto z(\cdot, v(\cdot))$ belongs to $C\Big(L^{b_1}(\Omega); L^{b_2}(\Omega)\Big)$.

Proposition 2.4 ([8, Proposition 1.4.1]). *If X is a reflexive Banach space, Y is a Banach space, $D \subset X$ is nonempty, closed and convex and $J \colon D \to Y$ is completely continuous, then J is compact.*

Now we formulate and prove the generalization of Proposition B.10 of Rabinowitz [12].

Proposition 2.5. *Let Ω be a bounded domain in $\mathbb{R}^N$ whose boundary is a smooth manifold and $p \in (1, +\infty)$. Let z satisfy:*

(z_1) $z \in C\big(\overline{\Omega} \times \mathbb{R}; \mathbb{R}\big)$,
(z_2) *there are constants a_1, $a_2 > 0$ and $\nu \in \big(0, p^* - 1\big)$ such that*

$$|z(x,\xi)| \leq \ a_1 + a_2|\xi|^{\nu}, \quad \forall (x,\xi) \in \overline{\Omega} \times \mathbb{R}.$$

If

$$Z(x,\xi) := \int_0^\xi z(x,t)\,dt \quad \text{and} \quad T(u) := \int_\Omega \left(\frac{1}{p}|\nabla u(x)|^p - Z(x,u(x))\right) dx,$$

then $T \in C^1\Big(W_0^{1,p}(\Omega); \mathbb{R}\Big)$ and

$$\big\langle T'(u), v \big\rangle = \int_\Omega \big(|\nabla u(x)|^{p-2}(\nabla u(x).\nabla v(x)) - z(x,u(x))v(x)\big)\,dx, \quad \forall v \in W_0^{1,p}(\Omega),$$

where $(\cdot.\cdot)$ denotes the scalar product in $\mathbb{R}^N$. Moreover,

$$J(u) := \int_\Omega Z(x,u(x))\,dx$$

is weakly continuous and $J'(u)$ is compact.

Proof. Let $X = W_0^{1,p}(\Omega)$ and denote by $\|\cdot\|$ the norm in X, by $\|\cdot\|_i$ the norm in $L^i(\Omega)$ and by c_i the constant of the embedding $W^{1,p}(\Omega) \subset L^i(\Omega)$. We will show that T is defined on X.

Let $u \in X$. We have

$$|J(u)| \le \int_\Omega \left| \int_0^{u(x)} z(x,t)\,dt \right| dx \le \int_\Omega \left(\int_0^{|u(x)|} \big(a_1 + a_2 t^\nu\big)\,dt \right) dx =$$
$$= a_1\|u\|_1 + \frac{a_2}{\nu+1}\|u\|_{\nu+1}^{\nu+1} \le a_1 c_1 \|u\| + \frac{a_2 c_{\nu+1}^{\nu+1}}{\nu+1}\|u\|^{\nu+1} < +\infty,$$

so then

$$|T(u)| \le \frac{1}{p}\|u\|^p + |J(u)| < +\infty.$$

In order to show that $T'(u)$ is defined on X for u fixed, let $v \in X$. Using the Hölder inequality, we obtain

$$\big|\langle T'(u), v\rangle\big| \le \int_\Omega |\nabla u(x)|^{p-1}|\nabla v(x)|\,dx + \int_\Omega |v(x)|\big(a_1 + a_2|u(x)|^\nu\big)\,dx \le$$
$$\le \left(\int_\Omega |\nabla u(x)|^p\,dx\right)^{\frac{p-1}{p}} \left(\int_\Omega |\nabla v(x)|^p\,dx\right)^{\frac{1}{p}} + a_1\|v\|_1 +$$
$$+ a_2 \left(\int_\Omega |v(x)|^{\nu+1}\,dx\right)^{\frac{1}{\nu+1}} \left(\int_\Omega |u(x)|^{\nu+1}\,dx\right)^{\frac{\nu}{\nu+1}} \le$$
$$\le \|u\|^{p-1}\|v\| + a_1 c_1\|v\| + a_2 c_{\nu+1}^{\nu+1}\|v\|\|u\|^\nu < +\infty.$$

The first term in T is C^1 and its Fréchet derivative is the first term in $T'(u)$ so we need to show that the second one belongs to $C^1(X;\mathbb{R})$. First we will show that J is Fréchet differentiable on X and then that $J'(u)$ is continuous. Put

$$\Upsilon := \big|Z(\cdot, u(\cdot) + v(\cdot)) - Z(\cdot, u(\cdot)) - z(\cdot, u(\cdot))v(\cdot)\big|.$$

We want to show that for all u, $v \in X$ and $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon, u, p) > 0$ such that

$$\left| J(u+v) - J(u) - \int_\Omega z(x, u(x))v(x)\,dx \right| \le \varepsilon\|v\| \tag{2.1}$$

provided $\|v\| \le \delta$. We have

$$\left| J(u+v) - J(u) - \int_\Omega z(x, u(x))v(x)\,dx \right| \le \int_\Omega \Upsilon(x)\,dx.$$

For any β, $\eta > 0$, we define

$$\Omega_1 := \big\{x \in \overline{\Omega} : |u(x)| \ge \beta\big\},$$
$$\Omega_2 := \big\{x \in \overline{\Omega} : |v(x)| \ge \eta\big\},$$
$$\Omega_3 := \big\{x \in \overline{\Omega} : |u(x)| \le \beta \ \wedge\ |v(x)| \le \eta\big\}.$$

Then

$$\int_{\Omega} \Upsilon(x)\,dx \leq \sum_{i=1}^{3} \int_{\Omega_i} \Upsilon(x)\,dx. \tag{2.2}$$

By the Mean Value Theorem, there exists $\theta \in (0,1)$ such that

$$Z(x,u(x)+v(x)) - Z(x,u(x)) = z(x,u(x)+\theta v(x))v(x).$$

Hence, using the Hölder inequality, we obtain

$$\begin{aligned}
&\int_{\Omega_1} \big|Z(x,u(x)+v(x)) - Z(x,u(x))\big|\,dx \leq \int_{\Omega_1} |v(x)|\big(a_1 + a_2(|u(x)|+|v(x)|)^{\nu}\big)\,dx \leq \\
&\quad \leq a_1\big|\Omega_1\big|^{\frac{p^*-1}{p^*}}\|v\|_{p^*} + a_2\big|\Omega_1\big|^{\frac{1}{\sigma}}\|v\|_{p^*}\Bigg(\int_{\Omega_1}(|u(x)|+|v(x)|)^{\nu+1}\,dx\Bigg)^{\frac{\nu}{\nu+1}} \leq \\
&\quad \leq a_1 c_{p^*}\big|\Omega_1\big|^{\frac{p^*-1}{p^*}}\|v\| + 2^{\frac{\nu^2}{\nu+1}}a_2 c_{p^*}\big|\Omega_1\big|^{\frac{1}{\sigma}}\|v\|\big(\|u\|_{\nu+1}^{\nu+1} + \|v\|_{\nu+1}^{\nu+1}\big)^{\frac{\nu}{\nu+1}} \leq \\
&\quad \leq a_3\|v\|\left(\big|\Omega_1\big|^{\frac{p^*-1}{p^*}} + \big|\Omega_1\big|^{\frac{1}{\sigma}}\big(\|u\|^{\nu+1} + \|v\|^{\nu+1}\big)^{\frac{\nu}{\nu+1}}\right),
\end{aligned}$$

where $\sigma \in (1,+\infty)$ is such that

$$\frac{1}{\sigma} = \frac{1}{\nu+1} - \frac{1}{p^*}$$

and $a_3 := c_{p^*}\max\Big\{a_1, 2^{\frac{\nu^2}{\nu+1}}a_2 c_{\nu+1}^{\nu}\Big\}$.

Using the Hölder inequality again, we have

$$\begin{aligned}
\int_{\Omega_1} |z(x,u(x))v(x)|\,dx &\leq a_1\big|\Omega_1\big|^{\frac{p^*-1}{p^*}}\|v\|_{p^*} + a_2\big|\Omega_1\big|^{\frac{1}{\sigma}}\|v\|_{p^*}\|u\|_{\nu+1}^{\nu} \leq \\
&\leq a_4\|v\|\left(\big|\Omega_1\big|^{\frac{p^*-1}{p^*}} + \big|\Omega_1\big|^{\frac{1}{\sigma}}\|u\|^{\nu}\right),
\end{aligned}$$

where $a_4 := c_{p^*}\max\big\{a_1, a_2 c_{\nu+1}^{\nu}\big\}$.

Due to the definition of Ω_1, we have the following estimates

$$\|u\| \geq c_p\|u\|_p \geq c_p\Bigg(\int_{\Omega_1}\beta^p\,dx\Bigg)^{\frac{1}{p}} = c_p\beta\big|\Omega_1\big|^{\frac{1}{p}},$$

and

$$\big|\Omega_1\big|^{\frac{1}{\sigma}} \leq \left(\frac{\|u\|}{c_p\beta}\right)^{\frac{p}{\sigma}} =: M_1, \qquad \big|\Omega_1\big|^{\frac{p^*-1}{p^*}} \leq \left(\frac{\|u\|}{c_p\beta}\right)^{\frac{p(p^*-1)}{p^*}} =: M_2,$$

where M_1, $M_2 \to 0$ as $\beta \to +\infty$. Then

$$\int\limits_{\Omega_1} \Upsilon(x)\,dx \le \ a_5\|v\| \left(M_2 + M_1\big(\|u\|^{\nu+1} + \|v\|^{\nu+1}\big)^{\frac{\nu}{\nu+1}}\right),$$

where $a_5 := \max\big\{a_3, a_4\big\}$.

Assume that $\delta \le 1$ and choose β so large that

$$a_5\big(M_2 + M_1\big(\|u\|^{\nu+1} + 1\big)\big) \le \ \frac{\varepsilon}{3}.$$

Hence

$$\int\limits_{\Omega_1} \Upsilon(x)\,dx \le \ \frac{\varepsilon}{3}\|v\|. \tag{2.3}$$

If we put $a_6 := 2\max\big\{a_1, a_2\big\}$ and use the same tools as above, we obtain

$$\int\limits_{\Omega_2} \Upsilon(x)\,dx \le a_6 \int\limits_{\Omega_2} |v(x)|\big(1 + (|u(x)| + |v(x)|)^{\nu}\big)dx \le$$

$$\le a_6\Bigg(\int\limits_{\Omega_2} |v(x)|^{\nu+1}\left(\frac{|v(x)|}{\eta}\right)^{p^*-(\nu+1)} dx\Bigg)^{\frac{1}{\nu+1}} \Bigg(\int\limits_{\Omega_2} \big(1 + (|u(x)| + |v(x)|)^{\nu}\big)^{\frac{\nu+1}{\nu}} dx\Bigg)^{\frac{\nu}{\nu+1}} \le$$

$$\le a_6\eta^{1-\frac{p^*}{\nu+1}}\|v\|_{p^*}^{\frac{p^*}{\nu+1}}\|1 + |u| + |v|\|_{\nu+1}^{\nu}.$$

If we put $a_7 := a_6 \max\big\{1, c_{\nu+1}^{\nu}\big\}c_{p^*}^{\frac{p^*}{\nu+1}}$, then

$$\int\limits_{\Omega_2} \Upsilon(x)\,dx \le \ a_7\eta^{1-\frac{p^*}{\nu+1}}\|v\|^{\frac{p^*}{\nu+1}}(1 + \|u\| + \|v\|)^{\nu}. \tag{2.4}$$

Since $Z \in C^1\big(\overline{\Omega} \times \mathbb{R}; \mathbb{R}\big)$, given any $\hat{\varepsilon}$, $\hat{\beta} > 0$, there exists $\hat{\eta} = \hat{\eta}\big(\hat{\varepsilon}, \hat{\beta}, p\big)$ such that

$$\big|Z(x, u(x) + v(x)) - Z(x, u(x)) - z(x, u(x))v(x)\big| \le \ \hat{\varepsilon}|v(x)|, \quad \forall x \in \Omega_3.$$

In particular, if $\hat{\beta} := \beta$ and $\eta \le \hat{\eta}$, then

$$\int\limits_{\Omega_3} \Upsilon(x)\,dx \le \ \hat{\varepsilon}\|v\|_1 \le c_1\hat{\varepsilon}\|v\|. \tag{2.5}$$

Choose $\hat{\varepsilon}$ so that $c_1\hat{\varepsilon} \le \frac{\varepsilon}{3}$. This determines $\hat{\eta}$, so take $\eta := \hat{\eta}$. Combining (2.2), (2.3), (2.4) and (2.5) yields

$$\int\limits_{\Omega} \Upsilon(x)\,dx \le \ \frac{2\varepsilon}{3}\|v\| + a_7\eta^{1-\frac{p^*}{\nu+1}}\|v\|^{\frac{p^*}{\nu+1}}(1 + \|u\| + \|v\|)^{\nu}.$$

Eventually choose δ so small that

$$a_7 \eta^{1-\frac{p^*}{\nu+1}} (2+\|u\|)^\nu \delta^{\frac{p^*}{\nu+1}-1} \leq \frac{\varepsilon}{3},$$

thereby obtaining (2.1).

In order to prove continuity of $J'(u)$, let $u_n \to u$ in X. Then $u_n \to u$ in $L^{\nu+1}(\Omega)$ by the Sobolev Embedding Theorem. By the Hölder inequality,

$$\begin{aligned}\big\|J'(u_n) - J'(u)\big\|_{X^*} &= \sup_{\|v\|\leq 1} \left| \int_\Omega \big(z\big(x, u_n(x)\big) - z(x, u(x))\big) v(x)\, dx \right| \leq \\ &\leq c_{\nu+1} \big\| z\big(\cdot, u_n(\cdot)\big) - z(\cdot, u(\cdot))\big\|_{\frac{\nu+1}{\nu}}.\end{aligned}$$

From hypothesis (z_2), we have

$$|z(x,\xi)| \leq a_1 + a_2 |\xi|^{\frac{\alpha\nu}{\alpha}}, \quad \forall x \in \overline{\Omega}\ \ \forall \xi \in \mathbb{R}\ \ \forall \alpha \geq 1.$$

Choosing $\alpha := \frac{\nu+1}{\nu}$ and using Proposition 2.3, we obtain that the map $u \mapsto z(\cdot, u(\cdot))$ belongs to $C\Big(L^{\nu+1}(\Omega); L^{\frac{\nu+1}{\nu}}(\Omega)\Big)$, so $J'\big(u_n\big) \to J'(u)$ as $n \to +\infty$. We proved that $J \in C^1(X;\mathbb{R})$.

To prove that J is weakly continuous, let $u_n \rightharpoonup u$ in X. By the Sobolev Embedding Theorem, $u_n \to u$ in $L^{\nu+1}(\Omega)$. Then using the Mean Value Theorem, hypothesis (z_2), the Hölder inequality and the Sobolev Embedding Theorem, we have

$$\begin{aligned}\big|J\big(u_n\big) - J(u)\big| &\leq \int_\Omega \big|u_n(x) - u(x)\big| \Big(a_1 + a_2 \big|u(x) + \theta\big(u_n(x) - u(x)\big)\big|^\nu\Big)\, dx \leq \\ &\leq a_1 \big\|u_n - u\big\|_1 + a_2 \big\|u_n - u\big\|_{\nu+1} \big\|u + \theta\big(u_n - u\big)\big\|_{\nu+1}^\nu \leq \\ &\leq \big\|u_n - u\big\|_{\nu+1} \bigg(a_1 |\Omega|^{\frac{\nu}{\nu+1}} + a_2 \left(c_{\nu+1}\|u\| + \theta \big\|u_n - u\big\|_{\nu+1} \right)^\nu \bigg)\end{aligned}$$

and hence $J\big(u_n\big) \to J(u)$ as $n \to +\infty$.

The last step is to show that $J'(u)$ is compact. Let $v_n \rightharpoonup v$ in X. Then $v_n \to v$ in $L^{\nu+1}(\Omega)$. Using the same tools as above, we obtain

$$\begin{aligned}\big|\big\langle J'(u), v_n \big\rangle - \big\langle J'(u), v \big\rangle\big| &\leq \int_\Omega \big|v_n(x) - v(x)\big| \big(a_1 + a_2 |u(x)|^\nu\big)\, dx \leq \\ &\leq \big\|v_n - v\big\|_{\nu+1} \Big(a_1 |\Omega|^{\frac{\nu}{\nu+1}} + a_2 c_{\nu+1}^\nu \|u\|^\nu \Big)\end{aligned}$$

and hence $\big\langle J'(u), v_n \big\rangle \to \big\langle J'(u), v \big\rangle$ as $n \to +\infty$. This means that $J'(u)$ is completely continuous so it is compact by Proposition 2.4. □

3. PROOF OF THE MAIN RESULT

In this section we prove Theorem 1.1.

Let $I = [0, +\infty)$ and $X = W_0^{1,p}(\Omega)$. The norm in X is defined by $\|u\| := \left(\int_\Omega |\nabla u(x)|^p \, dx \right)^{\frac{1}{p}}$. We put

$$J_1(u) := \int_\Omega F(u(x))\, dx, \qquad J_2(u) := \int_\Omega G(u(x))\, dx$$

for all $u \in X$. From (1.4), there exist $\eta \in (0, 1]$ and $c > 0$ such that

$$F(\xi) \leq c|\xi|^\gamma, \quad \forall \xi \in [-\eta, \eta].$$

By assumption (1.3), putting

$$c_1 := \max \left\{ c, \sup_{|\xi| > \eta} \frac{a\left(1 + |\xi|^s\right)}{|\xi|^\gamma} \right\},$$

we obtain

$$F(\xi) \leq c_1 |\xi|^\gamma, \quad \forall \xi \in \mathbb{R}.$$

By Sobolev the Embedding Theorem, there exists $c_2 > 0$ such that

$$\|v\|_\gamma \leq c_2 \|v\|_{1,p}, \quad \forall v \in W^{1,p}(\Omega)$$

as $\gamma < p^*$ and then

$$J_1(u) \leq c_1 c_2^\gamma \|u\|^\gamma, \quad \forall u \in X.$$

If $r > 0$ and $\|u\|^p \leq pr$, then

$$J_1(u) \leq c_3 r^{\frac{\gamma}{p}},$$

where $c_3 := c_1 c_2^\gamma p^{\frac{\gamma}{p}}$. Hence, we have

$$\lim_{r \to 0^+} \frac{\sup_{\|u\|^p \leq pr} J_1(u)}{r} = 0.$$

By assumption $\sup_{\xi \in \mathbb{R}} F(\xi) > 0$, so we can choose $w \in X \setminus \{0\}$ in such a way that $J_1(w) > 0$. We fix $r, \varepsilon > 0$ with $r < \frac{1}{p}\|w\|^p$, so that

$$\sup_{\|u\|^p \leq pr} J_1(u) \leq pr \frac{J_1(w)}{\|w\|^p} - \varepsilon$$

and fix $\delta > 0$ satisfying

$$\delta \left(\sup_{\|u\|^p \leq pr} \left|J_2(u)\right| + pr \frac{\left|J_2(w)\right|}{\|w\|^p} \right) < \varepsilon.$$

If we put

$$\sigma := \varepsilon - \delta\left(\sup_{\|u\|^p \le pr} \big|J_2(u)\big| + pr\frac{\big|J_2(w)\big|}{\|w\|^p}\right),$$

then simple calculations show that

$$\sup_{\|u\|^p \le pr} \big(J_1(u) + \mu J_2(u)\big) \le pr\frac{J_1(w) + \mu J_2(w)}{\|w\|^p} - \sigma, \quad \forall \mu \in [-\delta, \delta].$$

Further we fix $\mu \in [-\delta, \delta]$ and define functionals Φ, $\Psi \colon X \to \mathbb{R}$ and function $h \colon I \to \mathbb{R}$ by

$$\Phi(u) := \frac{1}{p}\|u\|^p, \qquad \Psi(u) := -\big(J_1(u) + \mu J_2(u)\big), \qquad h(\lambda) := \rho\lambda,$$

where ρ is a fixed number satisfying

$$\sup_{\|u\|^p \le pr} \big(J_1(u) + \mu J_2(u)\big) < \rho < pr\frac{J_1(w) + \mu J_2(w)}{\|w\|^p}.$$

The functional Φ is convex and continuous so it is sequentially weakly lower semicontinuous (see [19, Proposition 41.8]). Due to (1.2) and Proposition 2.5, the functional Ψ is continuously Gâteaux differentiable with compact Gâteaux derivative. For each u, $v \in X$ one has

$$\big\langle \Phi'(u), v \big\rangle = \int_\Omega \big|\nabla u(x)\big|^{p-2} (\nabla u(x).\nabla v(x))\, dx,$$

$$\big\langle \Psi'(u), v \big\rangle = -\int_\Omega (f(u(x)) + \mu g(u(x))) v(x)\, dx,$$

so the weak solutions of the problem

$$\begin{cases} -\Delta_p u = \lambda(f(u) + \mu g(u)) \text{ in } \Omega, \\ u|_{\partial\Omega} = 0 \end{cases}$$

are the critical points in X of the functional $\Phi + \lambda\Psi$.

We will show that

$$\lim_{\|u\| \to +\infty} (\Phi(u) + \lambda\Psi(u)) = +\infty, \quad \forall \lambda \in I. \tag{3.1}$$

Let $u \in X$. Since $s \in [1, p)$ and (1.3) holds, we have the following estimate:

$$-\Psi(u) \le \int_\Omega \big(F(u(x)) + |\mu||G(u(x))|\big)\, dx \le a(1 + |\mu|) \int_\Omega \big(1 + |u(x)|^s\big)\, dx \le$$
$$\le a(1 + |\mu|)\left(|\Omega| + \left(|\Omega|^{\frac{1}{s} - \frac{1}{p}} \|u\|_p\right)^s\right) \le a(1 + |\mu|)\left(|\Omega| + |\Omega|^{\frac{p-s}{p}} \|u\|^s\right).$$

Hence

$$\Phi(u)+\lambda\Psi(u)\geq \frac{1}{p}\|u\|^p-\lambda a(1+|\mu|)\left(|\Omega|+|\Omega|^{\frac{p-s}{p}}\|u\|^s\right),$$

which yields (3.1) while $\|u\|\to+\infty$.

Using Proposition 2.2 with $x_0=0$ and $x_1=w$, we obtain

$$\sup_{\lambda\geq 0}\inf_{u\in X}(\Phi(u)+\lambda(\rho+\Psi(u)))<\inf_{u\in X}\sup_{\lambda\geq 0}(\Phi(u)+\lambda(\rho+\Psi(u))),$$

which is equivalent to

$$\sup_{\lambda\in I}\inf_{u\in X}(\Phi(u)+\lambda\Psi(u)+h(\lambda))<\inf_{u\in X}\sup_{\lambda\in I}(\Phi(u)+\lambda\Psi(u)+h(\lambda)).$$

All the assumptions of Theorem 2.1 are satisfied so there exists an open interval $\Lambda\subset I$ such that, for each $\lambda\in\Lambda$, problem (1.1) has at least three weak solutions in X whose norms are less than ρ. □

4. REMARKS

In this section we formulate some remarks which show that the assumptions of Theorem 1.1 cannot be omitted.

Remark 4.1. The condition $\sup_{\xi\in\mathbb{R}}F(\xi)>0$ is essential. Suppose that this condition does not hold. Then taking $f=g=0$, problem (1.1) has only a trivial solution.

Remark 4.2. We cannot drop assumption (1.4). Suppose that $f=1$ and $g=0$. Then we can take $a=1$. Since $\gamma>p>1$, then

$$\limsup_{\xi\to 0}\frac{F(\xi)}{|\xi|^\gamma}=+\infty.$$

Let $\lambda\neq 0$ (otherwise we have the same situation as in Remark 4.1). Taking $N=1$, $p=4$, $\Omega=(A,B)$, where $A<B$, we obtain that the function

$$u(x)=-\frac{3}{8}\sqrt[3]{\frac{\lambda}{2}}\left((A+B-2x)^{\frac{4}{3}}-(B-A)^{\frac{4}{3}}\right),\qquad \forall x\in(A,B),$$

is the unique solution of the problem

$$\begin{cases}-3(u')^2u''=\lambda \text{ in } (A,B),\\ u(A)=u(B)=0.\end{cases}$$

Remark 4.3. Theorem 1.1 does not hold, in general, for any $\mu\in\mathbb{R}$. If f satisfies all the assumptions of Theorem 1.1, then taking $\mu=-1$ and $g=f$, we have the situation from Remark 4.1.

5. EXAMPLES

In this section we give some examples of problems to which one can use Theorem 1.1.

Example 5.1. Let $p = 3$, $N = 1$ and let Ω be an open and bounded interval. Consider functions $f(t) = 8t^3 \cos t^2$ and $g(t) = \sqrt[3]{t+1}$. Easy calculations show that these functions satisfy the assumptions of Theorem 1.1 with constants $a = 8$, $q = 3$, $s = 2$ and $\gamma = 4$ so there exists $\delta > 0$ such that, for each $\mu \in [-\delta, \delta]$, there exist $\rho > 0$ and an open interval $\Lambda \subset [0, +\infty)$ such that, for each $\lambda \in \Lambda$, the problem

$$\begin{cases} -\operatorname{div}\big(|\nabla u|\nabla u\big) = \lambda\Big(8u^3 \cos u^2 + \mu\sqrt[3]{u+1}\Big) \text{ in } \Omega, \\ u|_{\partial\Omega} = 0 \end{cases}$$

has at least three weak solutions in $W_0^{1,3}(\Omega)$ whose norms are less than ρ.

Example 5.2. Let $p = \frac{12}{7}$, $N = 3$ and $\Omega \subset \mathbb{R}^3$ be open and bounded. Consider functions

$$f(t) = \frac{4t^3}{t^4+1}, \qquad g(t) = \frac{t^2 \cos\sqrt{t^2+1}}{\sqrt{t^2+1}} + \sin\sqrt{t^2+1}.$$

Then f and g satisfy the assumptions of Theorem 1.1 with constants $a = 4$, $q = s = 1$ and $\gamma = 3$ so there exists $\delta > 0$ such that, for each $\mu \in [-\delta, \delta]$, there exist $\rho > 0$ and an open interval $\Lambda \subset [0, +\infty)$ such that, for each $\lambda \in \Lambda$, the problem

$$\begin{cases} -\operatorname{div}\Big(|\nabla u|^{-\frac{2}{7}}\nabla u\Big) = \lambda\left(\dfrac{4u^3}{u^4+1} + \mu\left(\dfrac{u^2\cos\sqrt{u^2+1}}{\sqrt{u^2+1}} + \sin\sqrt{u^2+1}\right)\right) \text{ in } \Omega, \\ u|_{\partial\Omega} = 0 \end{cases}$$

has at least three weak solutions in $W_0^{1,\frac{12}{7}}(\Omega)$ whose norms are less than ρ.

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