RANK-ONE PERTURBATION OF TOEPLITZ OPERATORS AND REFLEXIVITY

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Abstract. It was shown that rank-one perturbation of the space of Toeplitz operators preserves 2-hyperreflexivity.

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1. INTRODUCTION

Let $\mathcal{H}$ be a Hilbert space. By $\mathcal{B}(\mathcal{H})$ we denote the algebra of all bounded linear operators on $\mathcal{H}$.

It is well known that the space of trace class operators τc is a predual to $\mathcal{B}(\mathcal{H})$ with the dual action $\langle A, f\rangle = tr(Af)$, for $A \in \mathcal{B}(\mathcal{H})$ and $f \in \tau c$. The trace norm in τc will be denoted by $\|\cdot\|_1$. Denote by F_k the set of operators of rank at most k. Every rank-one operator may be written as $x \otimes y$, for $x, y \in \mathcal{H}$, and $(x \otimes y)z = \langle z, y\rangle x$ for $z \in \mathcal{H}$. Moreover, $tr(T(x \otimes y)) = \langle Tx, y\rangle$.

Let $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ be a subspace (when we write subspace we mean a norm closed linear manifold). By $d(T, \mathcal{M})$ we will denote the standard distance from an operator T to a subspace $\mathcal{M}$, i.e., $d(T, \mathcal{M}) = \inf\{\|T - M\| : M \in \mathcal{M}\}$. It is known that when $\mathcal{M}$ is weak* closed $d(T, \mathcal{M}) = \sup\{|tr(Tf)| : f \in \mathcal{M}_\perp, \ \|f\|_1 \leq 1\}$, where $\mathcal{M}_\perp$ denotes the preannihilator of $\mathcal{M}$.

Recall that *the reflexive closure* of a subspace $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ is given by

$$\operatorname{ref} \mathcal{M} = \{T \in \mathcal{B}(\mathcal{H}) : Tx \in [\mathcal{M}x] \text{ for all } x \in \mathcal{H}\},$$

where $[\cdot]$ denotes the norm-closure. A subspace $\mathcal{M}$ is called *reflexive* if $\mathcal{M} = \operatorname{ref} \mathcal{M}$. Due to Longstaff [14] we know that when $\mathcal{M}$ is a weak* closed subspace of $\mathcal{B}(\mathcal{H})$, then $\mathcal{M}$ is reflexive if and only if $\mathcal{M}_\perp$ is a closed linear span of the set of all operators of rank one contained in $\mathcal{M}_\perp$ (i.e., $\mathcal{M}_\perp = [\mathcal{M}_\perp \cap F_1]$). A subspace $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ is called *k-reflexive* if $\mathcal{M}^{(k)} = \{M^{(k)} : M \in \mathcal{M}\}$ is reflexive in $B(\mathcal{H}^{(k)})$, where

$M^{(k)} = M \oplus \cdots \oplus M$ and $\mathcal{H}^{(k)} = \mathcal{H} \oplus \cdots \oplus \mathcal{H}$. Kraus and Larson [12, Theorem 2.1] proved that a weak* closed subspace $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ is k-reflexive if and only if $\mathcal{M}_\perp$ is a closed linear span of rank-k operators contained in $\mathcal{M}_\perp$ (i.e., $\mathcal{M}_\perp = [\mathcal{M}_\perp \cap F_k]$).

In [2] Arveson defines an algebra $\mathcal{A}$ as *hyperreflexive* if there is a constant a such that $d(T, \mathcal{A}) \leq a \sup\{\|P^\perp T P\| : P \in Lat\mathcal{A}\}$ for all $T \in \mathcal{B}(\mathcal{H})$. In [11] this definition was generalized to subspaces of operators. A subspace $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ is called *hyperreflexive* if there is a constant a such that

$$d(T, \mathcal{M}) \leq a \sup\{\|Q^\perp T P\| : P, Q \text{ are projections and } Q^\perp \mathcal{M} P = 0\}$$

for all $T \in \mathcal{B}(\mathcal{H})$. As it was shown in [12] the supremum on the right hand side is equal to $\sup\{|\langle T, g \otimes h\rangle| : g \otimes h \in \mathcal{M}_\perp, \|g \otimes h\|_1 \leq 1\}$.

Recall after [10] the definition of k-hyperreflexivity. Let $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ be a subspace. For any $T \in \mathcal{B}(\mathcal{H})$ denote

$$\alpha_k(T, \mathcal{M}) = \sup\{|tr(Tf)| : f \in \mathcal{M}_\perp \cap F_k, \|f\|_1 \leq 1\}.$$

A subspace $\mathcal{M}$ is called *k-hyperreflexive* if there is $a > 0$ such that for any $T \in \mathcal{B}(\mathcal{H})$ the following inequality holds:

$$d(T, \mathcal{M}) \leq a\, \alpha_k(T, \mathcal{M}). \tag{1.1}$$

Let $\kappa_k(\mathcal{M})$ be the infimum of the collection of all constants a such that inequality (1.1) holds, then $\kappa_k(\mathcal{M})$ is a constant of k-hyperreflexivity. Operator T is *k-hyperreflexive* if the WOT closed algebra generated by T and identity is k-hyperreflexive.

When $k = 1$ the definition above coincides with the definition of hyperreflexivity and the letter k will be omitted.

2. REFLEXIVITY OF PERTURBATED TOEPLITZ OPERATORS

Let $\mathbb{T}$ be the unit circle on the complex plane $\mathbb{C}$. Denote $L^2 = L^2(\mathbb{T}, m)$ and $L^\infty = L^\infty(\mathbb{T}, m)$, where m is the normalized Lebesgue measure on $\mathbb{T}$. Let H^2 be the Hardy space corresponding to L^2 and P_{H^2} be a projection from L^2 onto H^2. For each $\phi \in L^\infty$ we define $T_\phi : H^2 \to H^2$ by $T_\phi f = P_{H^2}(\phi f)$ for $f \in H^2$. Operator T_ϕ is called a *Toeplitz operator* and $\mathcal{T}$ will denote the space of all Toeplitz operators.

The unilateral shift S can be realized as the multiplication operator by independent variable T_z. Moreover, $\mathcal{T} = \{T_\phi : \phi \in L^\infty\} = \{A : T_z^* A T_z = A\}$ ([9, Corollary 1 to Problem 194]). Hence $\mathcal{T}$ is weak* closed.

Let $\{e_j\}_{j\in\mathbb{N}}$ be the usual basis in H^2. Denote by $\mathcal{M}_{lm}$ the subspace $\mathcal{T} + \mathbb{C}(e_l \otimes e_m)$. In [4, Theorem 3.1] the authors proved that the space of all Toeplitz operators is not reflexive but it is 2-reflexive. We will show that the subspace $\mathcal{M}_{lm}$ has the same properties.

Proposition 2.1. *The subspace $\mathcal{M}_{lm}$ is not reflexive but it is 2-reflexive.*

Proof. Notice that $(\mathcal{M}_{lm})_\perp = \mathcal{T}_\perp \cap (e_l \otimes e_m)_\perp$. Since $\mathcal{T}_\perp$ contains no nonzero rank-one operators, then $\mathcal{M}_{lm}$ is not reflexive.

Notice that

$$\mathcal{T}_\perp = \text{span}\{e_i \otimes e_j - Se_i \otimes Se_j : i, j = 1, 2, \dots\},$$

where S is the unilateral shift. Therefore,

$$(\mathcal{M}_{lm})_\perp = \text{span}\{e_i \otimes e_j - Se_i \otimes Se_j : i, j = 1, 2, \dots, (i, j) \neq (l, m) \\ \text{and } (i+1, j+1) \neq (l, m)\}.$$

Hence $\mathcal{M}_{lm}$ is 2-reflexive. □

Recall after [5] the following definition.

Definition 2.2. Subspace $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ has *property* $\mathbb{A}_{1/k}$ if $\mathcal{M}$ is weak* closed and for any weak* continuous functional ϕ on $\mathcal{M}$ there is $g \in F_k$ such that $\phi(M) = tr(Mg)$ for $M \in \mathcal{M}$.

Proposition 2.3. *The subspace $\mathcal{M}_{lm} = \mathcal{T} + \mathbb{C}(e_l \otimes e_m)$ has property $\mathbb{A}_{1/4}$.*

Proof. Let $t \in \tau c$. Since $\mathcal{T}$ has property $\mathbb{A}_{1/2}$ ([10, Proposition 4.1]), there is $f \in F_2$ such that $(t - f) \in \mathcal{T}_\perp$. If $(t - f) \in (\mathbb{C}e_l \otimes e_m)_\perp$, then $(t - f) \in (\mathcal{M}_{lm})_\perp$. If $(t - f) \notin (\mathbb{C}e_l \otimes e_m)_\perp$, then $(t - f - \lambda e_l \otimes e_m + \lambda e_{l+1} \otimes e_{m+1}) \in (\mathcal{M}_{lm})_\perp$, where $\lambda = P_{\mathbb{C}e_l}(t - f)P_{\mathbb{C}e_m}$ and $P_{\mathbb{C}e_i}$ denotes the orthogonal projection on $\mathbb{C}e_i$. So $\mathcal{M}_{lm}$ has property $\mathbb{A}_{1/4}$. □

In [13] Larson proved that if $\mathcal{M}$ is k-reflexive, then any weak* closed subspace $\mathcal{L} \subset \mathcal{M}$ is k-reflexive if and only if $\mathcal{M}$ has property $\mathbb{A}_{1/k}$. It follows immediately from Proposition 2.1 and Proposition 2.3 that:

Corollary 2.4. *Every weak*-closed subspace of $\mathcal{M}_{lm} = \mathcal{T} + \mathbb{C}(e_l \otimes e_m)$ is 4-reflexive.*

On the other hand, due to [8] we know that the algebra of analytic Toeplitz operators is hyperreflexive. Moreover, the space of all Toeplitz operators $\mathcal{T}$ is 2-hyperreflexive and $\kappa_2(\mathcal{T}) \leq 2$ (see [10, 15]). We will show that the subspace $\mathcal{M}_{lm}$ is 2-hyperreflexive. In the proof we will use the projection $\pi : \mathcal{B}(H^2) \to \mathcal{T}$ constructed by Arveson in [1, Proposition 5.2], which has the property that for any $A \in \mathcal{B}(H^2)$ the operator $\pi(A)$ belongs to the weak* closed convex hull of the set $\{T^*_{z^n} A T_{z^n} : n \in \mathbb{N}\}$.

Proposition 2.5. *Subspace $\mathcal{M}_{lm} = \mathcal{T} + \mathbb{C}(e_l \otimes e_m)$ is 2-hyperreflexive with constant $\kappa_2(\mathcal{M}_{lm}) \leq 2$.*

Proof. Let $A \in \mathcal{B}(H^2)$. For $\lambda \in \mathbb{C}$ define $A_\lambda = A - \lambda e_l \otimes e_m$. Notice that for any $\lambda \in \mathbb{C}$

$$d(A, \mathcal{M}_{lm}) \leq \|A - \pi(A) - \lambda e_l \otimes e_m\| = \|A_\lambda - \pi(A_\lambda)\|.$$

Since the space of Toeplitz operators $\mathcal{T}$ is 2-hyperreflexive with constant at most 2, we have that

$$d(A_\lambda, \mathcal{T}) \leq \|A_\lambda - \pi(A_\lambda)\| \leq 2\alpha_2(A_\lambda, \mathcal{T}) \quad \text{(for details see [10])}.$$

To complete the proof it is enough to show that for any $A \in \mathcal{B}(H^2)$ there is $\lambda \in \mathbb{C}$ such that

$$\alpha_2(A_\lambda, \mathcal{T}) = \alpha_2(A, \mathcal{M}_{lm}). \tag{2.1}$$

Note that

$$\alpha_2(A_\lambda, \mathcal{T}) = \sup\{|tr(A_\lambda t)| : 2t = e_i \otimes e_j - e_{i+k} \otimes e_{j+k}, k \geq 1, i, j = 0, 1, 2, \dots\}.$$

If this supremum is realized by $2t = e_i \otimes e_j - e_{i+k} \otimes e_{j+k}$ for $(i,j) \neq (l,m)$ and $(i+k, j+k) \neq (l,m)$, then equality (2.1) holds. So, it is enough to consider the case when

$$\begin{aligned}\alpha_2(A_\lambda, \mathcal{T}) =& \sup\{|tr(A_\lambda t)| : 2t = e_l \otimes e_m - e_{l+k} \otimes e_{m+k}, k \geq \min\{-l, -m\}\} = \\ =& \sup\{\tfrac{1}{2}|a_{lm} - \lambda - a_{l+k,m+k}| : k \geq \min\{-l, -m\}\}.\end{aligned}$$

Suppose that $\alpha_2(A, \mathcal{M}_{lm}) = \beta > 0$. Note that for any λ we have $\beta \leq \alpha_2(A_\lambda, \mathcal{T})$. If we choose $\lambda = a_{lm} - a_{l+1,m+1}$, then

$$\alpha_2(A_\lambda, \mathcal{T}) = \sup\{\tfrac{1}{2}|a_{l+1,m+1} - a_{l+k,m+k}| : k \geq \min\{-l, -m\}\} \leq \beta.$$

Hence $\alpha_2(A_\lambda, \mathcal{T}) = \alpha_2(A, \mathcal{M}_{lm})$, which completes the proof. □

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