BOUNDARY VALUE PROBLEMS FOR SECOND ORDER DELAY DIFFERENTIAL EQUATIONS

Lidia Skóra

Abstract. We present some existence and uniqueness result for a boundary value problem for functional differential equations of second order.

Keywords: functional differential equation, existence, uniqueness, fixed point theorem.

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1. INTRODUCTION

The theory of functional differential equations has been investigated because of its important practical significance (see [1–3] and references therein). There are many books devoted to functional differential equations (see for example [4, 6]). For second order delay differential equations we refer the reader to the papers [5, 7–11] and the references therein.

In this paper we discuss the boundary value problem for functional differential equations of second order

$$x''(t) = f(t, x_t), \quad t \in J = [0, T],\ T > 0, \tag{1.1}$$

$$x_0 = \phi, \quad x'(T) = \beta x'(0),\ \beta > 1, \tag{1.2}$$

where $f : J \times C([-\tau, 0], \mathbb{R}) \to \mathbb{R}$ is a given function, $\phi \in C([-\tau, 0], \mathbb{R}), \tau > 0$. For any function $x \in C\left([-\tau, T], \mathbb{R}\right)$ and any $t \in J$, we let x_t denote the element of $C([-\tau, 0], \mathbb{R})$ defined by

$$x_t(s) = x(t + s), \qquad s \in [-\tau, 0].$$

Here $x_t(\cdot)$ represents the history of the state from time $t - \tau$, up to the present time t. Condition $x_0 = \phi$ implies that $x(s) = \phi(s), s \in [-\tau, 0]$. The supremum norm of $\phi \in C\left([-\tau, 0], \mathbb{R}\right)$ is defined by

$$\|\phi\|_0 = \sup_{-\tau \le s \le 0} |\phi(s)|.$$

Boundary value problems (1.1)–(1.2) constitute a very interesting and important class of problems. They include ordinary differential equations, differential equations with delayed arguments and integro-differential equations as special cases.

Equation (1.1) with different boundary conditions has been studied in [5]. Applying a quasilinearization technique two monotone sequences are constructed and sufficient conditions which imply the convergence of these sequences to the unique solution are given. Our paper is based on a fixed point theorem. The purpose of this paper is to present new existence and uniqueness result for equation (1.1) with conditions (1.2).

2. PRELIMINARIES

Let us start by defining what we mean by a solution of problem (1.1)-(1.2). Denote $C^\star = C\left([-\tau, T], \mathbb{R}\right) \cap C^2\left([0, T], \mathbb{R}\right)$.

Definition 2.1. A function $x \in C^\star$ is said to be a solution of (1.1)–(1.2) if x satisfies $x''(t) = f(t, x_t)$, $t \in J$ and the conditions (1.2).

We need the following auxiliary result.

Lemma 2.2. *Function $x \in C^\star$ is a solution of* (1.1)–(1.2), *where $f \in C(J \times C([-\tau, 0], \mathbb{R}), \mathbb{R})$ if and only if x is a solution of the following integral equation*

$$x(t) = \begin{cases} \phi(t), & t \in [-\tau, 0], \\ \phi(0) + \dfrac{t}{\beta - 1} \int\limits_0^T f(s, x_s)ds + \int\limits_0^t (t-s) f(s, x_s)ds, & t \in J. \end{cases}$$

Proof. If $x \in C^\star$ is a solution of (1.1)–(1.2), than we have

$$x''(t) = f(t, x_t), \quad t \in J. \tag{2.1}$$

Integration by parts gives

$$x(t) = x(0) + tx'(0) + \int\limits_0^t (t-s)x''(s)ds. \tag{2.2}$$

Differentiating (2.2), we get

$$x'(t) = x'(0) + \int\limits_0^t x''(s)ds.$$

Hence

$$x'(T) = x'(0) + \int\limits_0^T x''(s)ds.$$

Using the boundary condition we obtain

$$x'(0) + \int_0^T x''(s)ds = \beta x'(0).$$

Thus

$$x'(0) = \frac{1}{\beta - 1} \int_0^T x''(s)ds. \tag{2.3}$$

Equation (2.2), together with (2.1) and (2.3) implies

$$x(t) = \phi(0) + \frac{t}{\beta - 1} \int_0^T f(s, x_s)ds + \int_0^t (t - s)f(s, x_s)ds. \tag{2.4}$$

Conversely, if x is a solution of equation (2.4), then direct differentiation of (2.4) gives

$$\begin{aligned} x'(t) &= \frac{1}{\beta - 1} \int_0^T f(s, x_s)ds + \int_0^t f(s, x_s)ds, \\ x''(t) &= f(t, x_t) \quad t \in [0, T]. \end{aligned}$$

Hence

$$\begin{aligned} x'(0) &= \frac{1}{\beta - 1} \int_0^T f(s, x_s)ds, \\ x'(T) &= \frac{1}{\beta - 1} \int_0^T f(s, x_s) + \int_0^T f(s, x_s)ds = \frac{\beta}{\beta - 1} \int_0^T f(s, x_s)ds, \end{aligned}$$

which gives

$$x'(T) = \beta x'(0).$$

□

3. MAIN RESULT

We are now ready to state and prove the existence and uniqueness result for problem (1.1)–(1.2).

Theorem 3.1. *Assume that* $f \in C(J \times C([-\tau, 0], \mathbb{R}), \mathbb{R})$ *and there exists* $m \in L^1([0, T], \mathbb{R}_+)$ *such that*

$$|f(t, u) - f(t, \bar{u})| \leq m(t)\|u - \bar{u}\|_0 \tag{3.1}$$

for all $t \in [0,T], u, \bar{u} \in C([-\tau,0],\mathbb{R})$ *and*

$$M(T) < \frac{\ln \beta}{T}, \tag{3.2}$$

where $M(t) = \int_0^t m(r)dr$. *Then problem* (1.1)–(1.2) *has a unique solution* $x \in C^\star$.

Proof. For $x \in C([0,T],\mathbb{R})$, let

$$\|x\| = \max\left\{e^{-\gamma M(s)} \max\{|x(r)|, r \in [0,s]\}, s \in [0,T]\right\},$$

where

$$T < \gamma < \frac{\ln \beta}{M(T)}. \tag{3.3}$$

Such γ exists by assumption (3.2). We transform the problem (1.1)–(1.2) into a fixed point problem. Define an operator $N : C([0,T],\mathbb{R}) \to C([0,T],\mathbb{R})$ by

$$(Nx)(t) = \phi(0) + \frac{t}{\beta - 1}\int_0^T f(s,x_s)ds + \int_0^t (t-s)f(s,x_s)ds,$$

where $x_s(r) = x(s+r) = \phi(s+r)$ for $s+r \le 0$. For any $x, y \in C([0,T],\mathbb{R}), t \in J$, by (3.1) we have

$$\begin{aligned}
|(Nx)(t) - (Ny)(t)| \le{}& \frac{t}{\beta-1}\int_0^T |f(s,x_s) - f(s,y_s)|ds+ \\
&+ \int_0^t (t-s)|f(s,x_s) - f(s,y_s)|ds \le \\
&\le \frac{t}{\beta-1}\int_0^T m(s)\|x_s - y_s\|_0 ds + \int_0^t (t-s)m(s)\|x_s - y_s\|_0 ds \le \\
&\le \frac{T}{\beta-1}\int_0^T m(s)\|x_s - y_s\|_0 ds + \int_0^t tm(s)\|x_s - y_s\|_0 ds \le \\
&\le \frac{T}{\beta-1}\int_0^T m(s)\|x_s - y_s\|_0 ds + T\int_0^t m(s)\|x_s - y_s\|_0 ds.
\end{aligned}$$

Notice that if $s \in [0,\tau]$, then

$$\begin{aligned}
\|x_s - y_s\|_0 &= \sup_{r\in[-\tau,0]} \|x(s+r) - y(s+r)\| = \\
&= \max\{\|x(s+r) - y(s+r)\|, r \in [-s,0]\} = \\
&= \max\{\|x(r) - y(r)\|, r \in [0,s]\}.
\end{aligned}$$

If $s \in (\tau, T]$, then

$$\begin{aligned}\|x_s - y_s\|_0 = &\sup_{r\in[-\tau,0]} \|x(s+r) - y(s+r)\| = \\ &= \max\{\|x(s+r) - y(s+r)\|, r \in [s-\tau, s]\} \leq \\ &\leq \max\{\|x(r) - y(r)\|, r \in [0, s]\}.\end{aligned}$$

Therefore,

$$\begin{aligned}&|(Nx)(t) - (Ny)(t)| \leq \\ &\leq \frac{T}{\beta-1}\int_0^T m(s)e^{\gamma M(s)}e^{-\gamma M(s)} \max\{\|x(r) - y(r)\|, r \in [0, s]\}\, ds + \\ &+ T\int_0^t m(s)e^{\gamma M(s)}e^{-\gamma M(s)} \max\{\|x(r) - y(r)\|, r \in [0, s]\}\, ds \leq \\ &\leq \frac{T}{\beta-1}\|x-y\|\int_0^T m(s)e^{\gamma M(s)}ds + T\|x-y\|\int_0^t m(s)e^{\gamma M(s)}ds = \\ &= \frac{T}{\beta-1}\|x-y\|\frac{1}{\gamma}e^{\gamma M(s)}\Big|_0^T + T\|x-y\|\frac{1}{\gamma}e^{\gamma M(s)}\Big|_0^t = \\ &= \frac{T}{\beta-1}\|x-y\|\frac{e^{\gamma M(T)}-1}{\gamma} + T\|x-y\|\frac{e^{\gamma M(t)}-1}{\gamma} = \\ &= \frac{T}{\gamma}\|x-y\|\left(\frac{e^{\gamma M(T)}-1}{\beta-1} + e^{\gamma M(t)} - 1\right) = \\ &= \frac{T}{\gamma}\|x-y\|\left(\frac{e^{\gamma M(T)}-\beta}{\beta-1} + e^{\gamma M(t)}\right).\end{aligned}$$

It follows from (3.3) that

$$e^{\gamma M(T)} - \beta < 0,$$

therefore

$$|(Nx)(t) - (Ny)(t)| \leq \frac{T}{\gamma}\|x-y\|e^{\gamma M(t)}.$$

Thus

$$\begin{aligned}&\max_{s\in[0,t]}|(Nx)(s) - (Ny)(s)| \leq \frac{T}{\gamma}\|x-y\| \max_{s\in[0,t]} e^{\gamma M(s)} \leq \frac{T}{\gamma}\|x-y\|e^{\gamma M(t)}, \\ &e^{-\gamma M(t)} \max_{s\in[0,t]}|(Nx)(s) - (Ny)(s)| \leq \frac{T}{\gamma}\|x-y\|, \\ &\max_{t\in[0,T]}\left(e^{-\gamma M(t)} \max_{s\in[0,t]}|(Nx)(s) - (Ny)(s)|\right) \leq \frac{T}{\gamma}\|x-y\|,\end{aligned}$$

i.e.,

$$\|Nx - Ny\| \leq \frac{T}{\gamma}\|x - y\|.$$

Thus N is a contractive operator and by the Banach fixed point theorem, N has a unique fixed point $x \in C([0,T),\mathbb{R})$. The proof is complete. □

By Theorem 3.1, we can obtain the following result.

Theorem 3.2. *Assume that:*

(i) $f \in C\left(J \times C([-\tau,0],\mathbb{R}),\mathbb{R}\right)$,

(ii) *the Frechet derivative f_Φ exists, is a continuous linear operator satisfying*

$$|f_\Phi(t,\Phi)w| \leq L\|w\|_0 \tag{3.4}$$

for $t \in J, \Phi, w \in C([-\tau,0],\mathbb{R})$ and

$$0 \leq L < \frac{2(\beta-1)}{(\beta+1)T^2}. \tag{3.5}$$

Then problem (1.1)–(1.2) *has a unique solution $x \in C^*$.*

Proof. It follows from (3.4) that inequality (3.1) is satisfied with $m(t) := L$. In consequence $M(t) = Lt$. From inequality (3.5) we have

$$LT < \frac{2(\beta-1)}{(\beta+1)T}. \tag{3.6}$$

Note that

$$\frac{2(\beta-1)}{(\beta+1)} < \ln\beta \quad \text{for } \beta > 1.$$

This, together with (3.6) implies that

$$M(T) < \frac{\ln\beta}{T}.$$

As we see all conditions of Theorem 3.1 are fulfilled, so problem (1.1)–(1.2) has a unique solution $x \in C^*$. □

Example 3.3. Consider the following problem

$$\begin{cases} x''(t) = t^2 + 2\cos t \int\limits_{-1}^{0} x_t(s)ds, & t \in [0,\frac{1}{2}], \\ x(s) = \sin s,\ s \in [-1,0], & x'(\frac{1}{2}) = 3x'(0). \end{cases} \tag{3.7}$$

Here $T = \dfrac{1}{2}, \beta = 3, f(t,u) = t^2 + 2\cos t \int\limits_{-1}^{0} u(s)ds$. We have

$$|f(t,u) - f(t,\bar{u})| \leq 2\|u - \bar{u}\|_0,$$

for all $t \in [0,\frac{1}{2}], u, \bar{u} \in C([-1,0],\mathbb{R})$. Taking $m(t) = 2$, we see that assumption (3.2) is satisfied.

Conclusion: By Theorem 3.1, equation (3.7) has a unique solution.

Example 3.4. Consider the following problem

$$\begin{cases} x''(t) = t^2 x\left(t - \dfrac{1}{4}\right), & t \in \left[0, \dfrac{1}{2}\right], \\ x(s) = 0, \ s \in \left[-\dfrac{1}{4}, 0\right], & x'\left(\dfrac{1}{2}\right) = 2x'(0). \end{cases} \tag{3.8}$$

It is easy to prove that the conditions of Theorem 3.2 are true.
Conclusion: Problem (3.8) has a unique solution. Note that $x(t) = 0$, $t \in [-\frac{1}{4}, \frac{1}{2}]$, is a solution of (3.8), hence it is a unique solution of (3.8).

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Lidia Skóra

Cracow University of Technology
Faculty of Physics, Mathematics and Computer Science
ul. Warszawska 24, 31-155 Krakow, Poland

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