

ON THE ASYMPTOTIC BEHAVIOUR  
OF SOLUTIONS  
TO A LINEAR FUNCTIONAL EQUATION

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**Abstract.** We investigate the asymptotic behaviour at infinity of solutions of the equation

$$\varphi(x) = \int_S \varphi(x + M(s))\sigma(ds).$$

We show among others that, under some assumptions, any positive solution of the equation which is integrable on a vicinity of infinity or vanishes at  $+\infty$  tends on some sequence to zero faster than some exponential function, but it does not vanish faster than another such function.

**Keywords:** linear functional equations and inequalities, solutions with a constant sign, asymptotic behaviour of solutions.

**Mathematics Subject Classification:** 39B12, 39B22, 39B62, 26A12.

## 1. INTRODUCTION

The aim of this paper is to study the behaviour at infinity of solutions with a constant sign to the functional equation

$$\varphi(x) = \int_S \varphi(x + M(s))\sigma(ds). \tag{1.1}$$

More precisely, we shall compare at infinity these solutions with some exponential functions. The crucial role is played here by real roots of the characteristic equation

$$\int_S e^{\lambda M(s)}\sigma(ds) = 1. \tag{1.2}$$

Together with equation (1.1) we shall consider also the inequalities

$$\varphi(x) \geq \int_S \varphi(x + M(s))\sigma(ds) \quad (1.3)$$

and

$$\varphi(x) \leq \int_S \varphi(x + M(s))\sigma(ds). \quad (1.4)$$

Similar subject matter to this and some other equations and inequalities has been touched, among others, by B. Choczewski, L. Anczyk, M. Kuczma, R.O. Davies and A.J. Ostaszewski; see [7, Sec. 3.3], [4, Theorems 5 and 6], [12, Lemma 2.2.2] and [14].

Equation (1.1) was studied by many authors, see Part 4 of the survey paper [1] and Sections 4.4.2, 5.4 of the monograph [3]. The first who considered Equation (1.1) were G. Choquet and J. Deny assuming that  $M$  is the identity on a locally compact topological abelian group  $S$  and  $\sigma$  is a probabilistic measure defined on the  $\sigma$ -field generated by the family of all real continuous functions defined on  $S$ . They characterized all real continuous and bounded solutions of (1.1), see [10, Ch. VIII, §1]; cf. also [12, Ch. 2, Ch. 9] and [9]. In the case of a finite set  $S$ , M. Laczko [8] gave the form of all nonnegative solutions of (1.1) defined on  $\mathbb{R}$  which are Lebesgue measurable. If (1.2) has real roots, then any such a solution is a.e. equal to a linear combination of (at most two) functions of the form

$$p(x)e^{\lambda x}, \quad (1.5)$$

where  $p$  is a function constant on cosets with respect to  $\text{cl}\langle \text{supp}(\sigma \circ M^{-1}) \rangle$  and  $\lambda$  is a real root of (1.2); if (1.2) has no real roots, then the only such solution is the function a.e. equal to zero; see also [5, Ch. III]. In the general case all nonnegative and locally integrable solutions of (1.1) defined on  $\mathbb{R}$  are a.e. equal to a linear combination of functions of the form (1.5), see [12, Theorem 8.1.6].

Important theorems concerning inequality (1.3) obtained M. Pycia [11]. He investigated among others solutions of (1.3) such that

$$\liminf_{x \rightarrow -\infty} \varphi(x) \frac{e^{-\lambda x}}{|x|} \geq 0 \quad \text{and} \quad \liminf_{x \rightarrow +\infty} \varphi(x) \frac{e^{-\lambda x}}{|x|} \geq 0,$$

where  $\lambda$  is the only real root of (1.2), and showed that (under suitable assumptions) they are a.e. equal to  $\varphi(0)e^{\lambda x}$ .

Bounded and Borel solutions of (1.4) were studied recently by K. Baron and W. Jarczyk [2] in the case of probabilistic measure.

In the above mentioned papers a key point is the fact that solutions considered there have a constant sign on the whole coset (or on the subsemigroup generated by  $\text{supp}(\sigma \circ M^{-1})$ ), or have specified properties in both infinities. R.O. Davies and A.J. Ostaszewski [4] were the first authors who studied, without any assumptions on roots of (1.2), solutions of (1.1) having a constant sign on a vicinity of one of infinities. However, they considered only the case, where  $S$  consists of two elements.

They showed that the existence of such solutions having some additional properties implies the existence of real roots of (1.2) of a specified sign; concerning the general case see [13–15]. Note that there exist solutions of (1.1) positive on a half-line which have no constant sign on  $\mathbb{R}$ , cf. [14, Preliminaries].

At present we will consider solutions having a constant sign only on a vicinity of infinity showing among other things that their behaviour is connected with real roots of (1.2).

## 2. ASSUMPTIONS AND DEFINITIONS

In what follows  $(S, \Sigma, \sigma)$  is a measure space with a finite measure  $\sigma$  and  $M: S \rightarrow \mathbb{R}$  is a  $\Sigma$ -measurable bounded function with  $\sigma(M \neq 0) > 0$ . Moreover,

$$m := \sup\{|M(s)| : s \in S\}.$$

A nonempty set  $W \subset \mathbb{R}$  is called *invariant* if  $W + \langle M(S) \rangle \subset W$ , where  $\langle M(S) \rangle$  denotes the additive subgroup of  $\mathbb{R}$  generated by  $M(S)$ . By a *solution* of (1.1) (resp. (1.3), resp. (1.4)) we mean a real function  $\varphi$  defined on a set of the form  $(a, +\infty) \cap W$ , where  $a \in [-\infty, +\infty)$  and  $W \subset \mathbb{R}$  is invariant, such that for every  $x \in (a+m, +\infty) \cap W$  the integral  $\int_S \varphi(x + M(s))\sigma(ds)$  exists and (1.1) (resp. (1.3), resp. (1.4)) holds. By a *regular solution* we mean a solution defined on an interval, which is Borel measurable and Lebesgue integrable on every finite interval contained in its domain. By a *positive* (resp. *negative*) *solution* we mean a solution which is nonnegative (resp. nonpositive) and positive a.e. (resp. negative a.e.) on its domain. By a *solution with a constant sign* we mean a solution which is either positive or negative.

## 3. PRELIMINARIES

Considering real roots of (1.2) we shall make use of the following remark.

**Remark 3.1.** The function  $u: \mathbb{R} \rightarrow \mathbb{R}$  given by

$$u(\lambda) = \int_S e^{\lambda M(s)} \sigma(ds)$$

is smooth and strictly convex. In particular equation (1.2) has at most two real roots. Since

$$u(\lambda) = \int_{\{M < 0\}} e^{\lambda M(s)} \sigma(ds) + \sigma(M=0) + \int_{\{M > 0\}} e^{\lambda M(s)} \sigma(ds)$$

for  $\lambda \in \mathbb{R}$ , we have

$$u(-\infty) = +\infty \cdot \sigma(M < 0) + \sigma(M=0), \quad u(+\infty) = +\infty \cdot \sigma(M > 0) + \sigma(M=0).$$

Consequently,  $u$  increases if and only if  $M \geq 0$  a.e., and  $u$  decreases if and only if  $M \leq 0$  a.e.

In our further investigations we shall also rely on the following fact; cf. [11, Remark 2].

**Remark 3.2.** Assume  $\lambda$  is a real number. Then  $\varphi$  is a solution of (1.3) if and only if the function

$$\frac{\varphi(x)}{e^{\lambda x}} \quad (3.1)$$

is a solution of

$$\psi(x) \geq \int_S \psi(x + M(s))\tau(ds), \quad (3.2)$$

where  $\tau: \Sigma \rightarrow [0, +\infty)$  is given by

$$\tau(A) = \int_A e^{\lambda M(s)}\sigma(ds). \quad (3.3)$$

Clearly, for every  $A \in \Sigma$ ,  $\tau(A) > 0$  if and only if  $\sigma(A) > 0$ , and

$$\int_S e^{(\lambda_0 + \lambda)M(s)}\sigma(ds) = \int_S e^{\lambda_0 M(s)}\tau(ds)$$

for any number  $\lambda_0$ .

#### 4. ASYMPTOTIC BEHAVIOUR OF REGULAR SOLUTIONS

According to [13, Theorem 2], the existence of a positive regular solution of (1.3) implies the existence of a real root of (1.2). Here we compare the behaviour of solutions of (1.1) with  $e^{\gamma x}$ , where  $\gamma$  depends on the real root of (1.2).

The real roots of (1.2), if they exist, we shall denote by  $\lambda_1$  and  $\lambda_2$  assuming that if they are different then  $\lambda_1 < \lambda_2$ .

**Theorem 4.1.** *If  $\varphi: (a, +\infty) \rightarrow \mathbb{R}$  is a positive regular solution of (1.1), then*

$$\int_c^{+\infty} \frac{\varphi(x)}{e^{\lambda_1 x}} dx = +\infty \quad \text{for } c \geq a \quad (4.1)$$

and

$$\limsup_{x \rightarrow +\infty} \frac{\varphi(x)}{e^{\lambda_1 x}} > 0. \quad (4.2)$$

*Proof.* With  $\lambda = \lambda_1$  let  $\psi$  denote the function (3.1) and define  $\tau: \Sigma \rightarrow [0, +\infty)$  by (3.3). It follows from Remark 3.2 that  $\psi$  is a solution of

$$\psi(x) = \int_S \psi(x + M(s))\tau(ds).$$

Applying [13, Theorems 27 and 29] and Remark 3.2 we infer that  $\psi$  is not integrable on a half-line infinite from the right and due to [13, Corollary 36] and Remark 3.2 it does not vanish at  $+\infty$ .  $\square$

**Theorem 4.2.** *If  $\sigma(M < 0) > 0$  and  $\varphi: (a, +\infty) \rightarrow \mathbb{R}$  is a positive regular solution of (1.3), then (4.1) holds and*

$$\limsup_{x \rightarrow +\infty} \frac{\varphi(x)}{e^{\gamma x}} = +\infty \quad \text{for } \gamma < \lambda_1.$$

*Proof.* The property (4.1) we get, proceeding as before, from [13, Theorem 29].

Suppose that

$$\limsup_{x \rightarrow +\infty} \frac{\varphi(x)}{e^{\gamma x}} < +\infty$$

for some  $\gamma < \lambda_1$  and fix  $\lambda \in (\gamma, \lambda_1)$ . Then the function (3.1) vanishes at  $+\infty$  and according to Remark 3.2 it is a (regular) solution of (3.2) with  $\tau$  given by (3.3). Since  $\tau(S) = u(\lambda) \neq 1$ , it follows from [13, Remark 33] and Remark 3.2 that there is a real number  $\kappa < 0$  with  $u(\kappa + \lambda) = 1$ . This contradicts the fact that  $\kappa + \lambda < \lambda_1$ .  $\square$

**Theorem 4.3.** *If*

$$M \geq 0 \quad \text{a.e. and } \sigma(M=0) < 1, \tag{4.3}$$

*then every positive regular solution  $\varphi: (a, +\infty) \rightarrow \mathbb{R}$  of (1.4) has the properties (4.1) and (4.2).*

In the proof we will need two other facts.

**Lemma 4.4.** *Assume (4.3). If (1.4) has a positive regular solution integrable on a half-line infinite from the right, then (1.2) has a negative root.*

*Proof.* Let  $\varphi: (a, +\infty) \rightarrow \mathbb{R}$  be a positive, regular and integrable solution of (1.4) with a finite  $a$ . Applying [13, Lemma 4] we see that the function

$$\varphi(x) = \int_x^{+\infty} \varphi(y) dy, \quad x > a, \tag{4.4}$$

is a positive (regular) solution of (1.4) vanishing at  $+\infty$  and the existence of a negative root of (1.2) gives the following lemma.  $\square$

**Lemma 4.5.** *Assume (4.3). If (1.4) has a positive regular solution vanishing at infinity, then (1.2) has a negative root.*

*Proof.* Let  $\varphi: (a, +\infty) \rightarrow \mathbb{R}$  be a positive regular solution of (1.4) vanishing at infinity and  $a \in \mathbb{R}$ . Using [13, Lemma 25] we have

$$(1 - \sigma(S)) \int_{a+m}^{+\infty} \varphi(y) dy \leq \int_S \left( \int_{a+m+M(s)}^{a+m} \varphi(y) dy \right) \sigma(ds). \tag{4.5}$$

Since  $\int_{a+m+M(s)}^{a+m} \varphi(y)dy$  is a.e. nonpositive and negative on the set of positive measure, viz. on  $\{M>0\}$ , the right-hand-side of (4.5) is negative. Consequently  $\sigma(S) > 1$ , i.e.,  $u(0) > 1$ , and so (cf. Remark 3.1) (1.2) has no nonnegative root. But it follows from (4.3) that  $u(-\infty) < 1$  and thus (1.2) has a negative root.  $\square$

*Proof of Theorem 4.3.* According to Remark 3.2 we can assume that  $\lambda_1 = 0$ . Then (1.2) has no negative root and it is enough to use Lemmas 4.4 and 4.5.  $\square$

**Theorem 4.6.** *If  $\varphi: (a, +\infty) \rightarrow \mathbb{R}$  is a positive regular solution of (1.1), then*

$$\int_c^{+\infty} \frac{\varphi(x)}{e^{\gamma x}} dx < +\infty \quad \text{for any real numbers } c \geq a \text{ and } \gamma > \lambda_2 \quad (4.6)$$

and

$$\liminf_{x \rightarrow +\infty} \frac{\varphi(x)}{e^{\lambda_2 x}} < +\infty \quad \text{or} \quad \int_S M(s)e^{\lambda_2 M(s)} \sigma(ds) = 0. \quad (4.7)$$

*Proof.* Fix  $\gamma > \lambda_2$ . Supposing that the integral in (4.6) is infinite and applying Remark 3.2 and [13, Theorems 16 and 19] we infer that (1.2) has a root in interval  $[\gamma, +\infty)$ ; a contradiction.

For the proof of (4.7) we can assume that  $\lambda_2 = 0$ . Then (1.2) has no positive root and according to [13, Theorems 7 and 9] we have (4.7).  $\square$

**Theorem 4.7.** *If  $\sigma(M > 0) > 0$ , then every positive regular solution  $\varphi: (a, +\infty) \rightarrow \mathbb{R}$  of (1.3) has the properties (4.6) and (4.7).*

*Proof.* We proceed as before using now [13, Theorems 19 and 9].  $\square$

**Theorem 4.8.** *If*

$$M \leq 0 \text{ a.e.} \quad \text{and} \quad \sigma(M=0) < 1, \quad (4.8)$$

and  $\varphi: (a, +\infty) \rightarrow \mathbb{R}$  is a positive regular solution of (1.4), then (4.6) holds and

$$\liminf_{x \rightarrow +\infty} \frac{\varphi(x)}{e^{\lambda_2 x}} < +\infty.$$

In the proof we will need the following lemma.

**Lemma 4.9.** *Assume (4.8). If (1.4) has a positive regular solution which is not integrable on a half-line infinite from the right, then (1.2) has a nonnegative root.*

*Proof.* We can assume that  $u(0) \neq 1$ , i.e., that  $\sigma(S) \neq 1$ . Let  $\varphi: (a, +\infty) \rightarrow \mathbb{R}$  be a positive regular solution of (1.4) which is not integrable and  $a \in \mathbb{R}$ . Making use of [13, Lemma 3] it is easy to check that the function

$$\int_{a+m}^x \varphi(y)dy + \frac{1}{\sigma(S) - 1} \int_S \left( \int_{a+m+M(s)}^{a+m} \varphi(y)dy \right) \sigma(ds), \quad x > a, \quad (4.9)$$

is also a regular solution of (1.4). Clearly (4.9) has an infinite limit at  $+\infty$ . Applying now [15, Lemma 6] we infer that (1.2) has a positive root.  $\square$

*Proof of Theorem 4.8.* We proceed as in the proof of Theorem 4.6 using Lemma 4.9 and [15, Lemma 6].  $\square$

We show at present that we cannot replace the inequalities from Theorems 4.2, 4.3, 4.7, 4.8 by the opposite ones there. More precisely, we show that these inequalities have analytic solutions tending very fast to zero (in the first two cases) and to infinity (in the last two cases). The next two technical lemmas are very helpful.

**Lemma 4.10.** *Assume  $\varphi: (b, +\infty) \rightarrow (0, +\infty)$  is a decreasing function.*

(i) *If there is an  $r \in (0, +\infty)$  such that*

$$\limsup_{x \rightarrow +\infty} \frac{\varphi(x+r)}{\varphi(x)} < \sigma(M < -r), \tag{4.10}$$

*then there exists a real number  $a \geq b$  such that the function  $\varphi|_{(a, +\infty)}$  is a solution of (1.4).*

(ii) *If  $M \geq 0$  a.e. and there is an  $r \in (0, +\infty)$  such that  $\sigma(M \geq r) > 0$  and*

$$\limsup_{x \rightarrow +\infty} \frac{\varphi(x+r)}{\varphi(x)} < \frac{1 - \sigma(M < r)}{\sigma(M \geq r)}, \tag{4.11}$$

*then there exists a real number  $a \geq b$  such that the function  $\varphi|_{(a, +\infty)}$  is a solution of (1.3).*

*Proof.* Assuming (4.10) we can find a number  $a \geq b$  such that

$$\frac{\varphi(x+r)}{\varphi(x)} < \sigma(M < -r) \quad \text{for } x > a$$

and for any  $x > a + m$  we have

$$\int_S \varphi(x + M(s))\sigma(ds) \geq \sigma(M < -r)\varphi(x - r) > \varphi(x).$$

In the case of (4.11) take an  $a \geq b$  such that

$$\frac{\varphi(x+r)}{\varphi(x)} < \frac{1 - \sigma(M < r)}{\sigma(M \geq r)} \quad \text{for } x > a.$$

Since  $M \geq 0$  a.e., for any  $x > a + m$  we get then

$$\begin{aligned} \int_S \varphi(x + M(s))\sigma(ds) &= \int_{\{M < r\}} \varphi(x + M(s))\sigma(ds) + \int_{\{M \geq r\}} \varphi(x + M(s))\sigma(ds) \leq \\ &\leq \sigma(M < r)\varphi(x) + \sigma(M \geq r)\varphi(x+r) < \varphi(x). \end{aligned} \quad \square$$

**Lemma 4.11.** *Assume  $\varphi: (b, +\infty) \rightarrow (0, +\infty)$  is an increasing function.*

(i) *If there is an  $r \in (0, +\infty)$  such that*

$$\limsup_{x \rightarrow +\infty} \frac{\varphi(x)}{\varphi(x+r)} < \sigma(M > r), \tag{4.12}$$

*then there exists a real number  $a \geq b$  such that the function  $\varphi|_{(a, +\infty)}$  is a solution of (1.4).*

(ii) *If  $M \leq 0$  a.e. and there is an  $r \in (0, +\infty)$  such that  $\sigma(M \leq -r) > 0$  and*

$$\limsup_{x \rightarrow +\infty} \frac{\varphi(x)}{\varphi(x+r)} < \frac{1 - \sigma(M > -r)}{\sigma(M \leq -r)}, \tag{4.13}$$

*then there exists a real number  $a \geq b$  such that the function  $\varphi|_{(a, +\infty)}$  is a solution of (1.3).*

*Proof.* Assuming (4.12) we can find a number  $a \geq b$  such that

$$\frac{\varphi(x)}{\varphi(x+r)} < \sigma(M > r) \quad \text{for } x > a$$

and for any  $x > a + m$  we have

$$\int_S \varphi(x + M(s))\sigma(ds) \geq \sigma(M > r)\varphi(x+r) > \varphi(x).$$

In the case of (4.13) take an  $a \geq b$  such that

$$\frac{\varphi(x)}{\varphi(x+r)} < \frac{1 - \sigma(M > -r)}{\sigma(M \leq -r)} \quad \text{for } x > a.$$

Since  $M \leq 0$ , for any  $x > a + m$  we then get

$$\int_S \varphi(x + M(s))\sigma(ds) \leq \sigma(M \leq -r)\varphi(x-r) + \sigma(M > -r)\varphi(x) < \varphi(x). \quad \square$$

In our construction of the above mentioned analytic solutions we use a strictly increasing, convex and analytic solution  $A: (0, +\infty) \rightarrow (1, +\infty)$  of the equation

$$A(x+r) = e^{A(x)} - 1 \tag{4.14}$$

satisfying the condition

$$\lim_{x \rightarrow +\infty} \frac{A(x)}{\exp^n(x)} = +\infty \quad \text{for } n \in \mathbb{N}, \tag{4.15}$$

where  $\exp^n$  denotes  $n$ -th iterate of  $\exp$ . It is well known (see [6, p.174]) that for any  $r \in (0, +\infty)$  such a solution exists.



**Theorem 4.12.** *If  $\sigma(M < 0) > 0$ , then (1.4) has a strictly decreasing and analytic solution  $\varphi: (a, +\infty) \rightarrow (0, +\infty)$  such that*

$$\lim_{x \rightarrow +\infty} \varphi(x) \exp^n(x) = 0 \quad \text{for } n \in \mathbb{N}. \tag{4.16}$$

*Proof.* Take an  $r \in (0, +\infty)$  with  $\sigma(M < -r) > 0$ , a strictly increasing, analytic solution  $A: (0, +\infty) \rightarrow (1, +\infty)$  of (4.14) satisfying (4.15) and define  $\varphi: (0, +\infty) \rightarrow (0, 1)$  by

$$\varphi(x) = \frac{1}{A(x)}. \tag{4.17}$$

Clearly  $\varphi$  is a strictly decreasing and analytic function. Making use of (4.14), (4.15) we get

$$\lim_{x \rightarrow +\infty} \frac{\varphi(x+r)}{\varphi(x)} = \lim_{x \rightarrow +\infty} \frac{A(x)}{e^{A(x)} - 1} = 0 < \sigma(M < -r)$$

and (4.16). By virtue of the first part of Lemma 4.10, there exists an  $a \geq 0$  such that  $\varphi|_{(a, +\infty)}$  is a solution of (1.4). □

**Theorem 4.13.** *If (4.3) holds, then (1.3) has a strictly decreasing and analytic solution  $\varphi: (a, +\infty) \rightarrow (0, +\infty)$  with (4.16).*

*Proof.* According to (4.3) we can choose a positive real  $r$  so that

$$\sigma(M < r) < 1 \quad \text{and} \quad \sigma(M \geq r) > 0.$$

Let  $A: (0, +\infty) \rightarrow (1, +\infty)$  be a strictly increasing and analytic solution of (4.14) satisfying (4.15). Using the second part of Lemma 4.10 we see that the desired solution is a restriction of  $\varphi: (0, +\infty) \rightarrow (0, 1)$  given by (4.17). □

**Remark 4.14.** For any Lebesgue measurable function  $\varphi: \mathbb{R} \rightarrow (0, +\infty)$  satisfying (4.16) there is a real number  $a$  such that

$$\int_a^{+\infty} \varphi(x) \exp^n(x) dx < +\infty \quad \text{for } n \in \mathbb{N}.$$

*Proof.* It is enough to prove that

$$\frac{\exp^n(x)}{\exp^{n+1}(x)} \leq e^{-x} \quad \text{for } x \geq \ln 2 \text{ and } n \in \mathbb{N}.$$

We omit an easy inductive proof of this property. □

**Theorem 4.15.** *If  $\sigma(M > 0) > 0$ , then (1.4) has a strictly increasing, convex and analytic solution  $\varphi: (a, +\infty) \rightarrow \mathbb{R}$  such that*

$$\lim_{x \rightarrow +\infty} \frac{\varphi(x)}{\exp^n(x)} = +\infty \quad \text{for } n \in \mathbb{N}. \tag{4.18}$$

*Proof.* Take an  $r \in (0, +\infty)$  with  $\sigma(M > r) > 0$  and a strictly increasing, convex and analytic solution  $A: (0, +\infty) \rightarrow (1, +\infty)$  of (4.14) satisfying (4.15). By virtue of the first part of Lemma 4.11 there is an  $a \geq 0$  such that  $A|_{(a, +\infty)}$  is a solution of (1.4).  $\square$

**Theorem 4.16.** *If (4.8) holds, then (1.3) has a strictly increasing, convex and analytic solution  $\varphi: (a, +\infty) \rightarrow \mathbb{R}$  with (4.18).*

*Proof.* According to (4.8) we can choose a positive real  $r$  so that

$$\sigma(M > -r) < 1 \quad \text{and} \quad \sigma(M \leq -r) > 0.$$

Taking now a strictly increasing, convex and analytic solution  $A: (0, +\infty) \rightarrow (1, +\infty)$  of (4.14) satisfying (4.15) and applying the second part of Lemma 4.11 we obtain the desired solution of (1.3).  $\square$

### 5. ASYMPTOTIC BEHAVIOUR OF INCREASING SOLUTIONS

Here we show that if  $\sigma(S) \neq 1$ , then near infinity the graph of any increasing solution of (1.1) is situated over the graph of some exponential function with infinite limit at  $+\infty$ . Let

$$\rho := \frac{\ln \sigma(S)}{m}.$$

**Theorem 5.1.** *If  $\varphi: (a, +\infty) \cap W \rightarrow (0, +\infty)$  is an increasing solution of (1.1), then*

$$\frac{\varphi(x)}{e^{|\rho|x}} \geq \min \left\{ \sigma(S), \frac{1}{\sigma(S)} \right\} \cdot \sup \left\{ \frac{\varphi(y)}{e^{|\rho|y}} : y \in (a, x] \cap W \right\} \quad \text{for } x \in (a, +\infty) \cap W,$$

*and, moreover, either  $|\rho|$  is a root of (1.2) or*

$$\lim_{x \rightarrow +\infty} \frac{\varphi(x)}{e^{|\rho|x}} = +\infty.$$

This result is an immediate consequence of theorems concerning inequalities given below, viz. Corollary 5.4 and Theorems 5.9, 5.2.

**Theorem 5.2.** *If  $\sigma(S) > 1$  and  $\varphi: (a, +\infty) \cap W \rightarrow \mathbb{R}$  is a solution of (1.3) bounded from below by a positive constant  $C$ , then  $a$  is a real number and*

$$\frac{\varphi(x)}{e^{\rho x}} \geq \frac{C}{\sigma(S)e^{\rho a}} \quad \text{for } x \in (a, +\infty) \cap W; \tag{5.1}$$

*moreover, either  $\rho$  is a root of (1.2) or*

$$\lim_{x \rightarrow +\infty} \frac{\varphi(x)}{e^{\rho x}} = +\infty. \tag{5.2}$$

*Proof.* According to [14, Lemma 1] we have

$$\varphi(x) \geq C\sigma(S)^n \quad \text{for } n \in \mathbb{N} \cup \{0\} \text{ and } x \in (a + nm, +\infty) \cap W.$$

In particular  $a$  is finite. Fix  $x \in (a, +\infty) \cap W$  and put

$$N = \max\{n \in \mathbb{N} \cup \{0\} : x > a + nm\}.$$

Then

$$\frac{\sigma(S)^N}{e^{\rho x}} = e^{(N - \frac{x}{m}) \ln \sigma(S)} \geq e^{-(1 + \frac{a}{m}) \ln \sigma(S)} = \frac{1}{\sigma(S)e^{\rho a}}$$

and

$$\varphi(x) \geq C\sigma(S)^N,$$

whence (5.1) follows.

Assume now that

$$\liminf_{x \rightarrow +\infty} \frac{\varphi(x)}{e^{\rho x}} < +\infty.$$

Since, due to (5.1), we have

$$0 < \liminf_{x \rightarrow +\infty} \frac{\varphi(x)}{e^{\rho x}},$$

it results from [14, Theorem 2(i)] that  $u(\rho) \leq 1$ . Moreover,

$$u(\rho) = \int_S e^{\rho M(s)} \sigma(ds) \geq \int_S e^{-\rho m} \sigma(ds) = 1,$$

i.e.,  $\rho$  is a root of (1.2). □

The following remark shows that in most cases we have (5.2).

**Remark 5.3.** If  $\sigma(S) \neq 1$ , then  $\rho$  is a root of (1.2) if and only if  $M = -m$  a.e. (and in this case  $\rho$  is the only real root of (1.2)), and  $-\rho$  is a root of (1.2) if and only if  $M = m$  a.e. (and in this case  $-\rho$  is the only real root of (1.2)).

From the above theorem we draw two conclusions concerning increasing solutions as well as solutions which are not integrable on a half-line infinite from the right.

**Corollary 5.4.** *If  $\sigma(S) > 1$  and  $\varphi: (a, +\infty) \cap W \rightarrow (0, +\infty)$  is an increasing solution of (1.3), then*

$$\frac{\varphi(x)}{e^{\rho x}} \geq \frac{1}{\sigma(S)} \sup \left\{ \frac{\varphi(y)}{e^{\rho y}} : y \in (a, x] \cap W \right\} \quad \text{for } x \in (a, +\infty) \cap W.$$

*Proof.* Fix  $x \in (a, +\infty) \cap W$  and  $y \in (a, x] \cap W$ . Since  $\varphi|_{(y, +\infty) \cap W}$  is a solution of (1.3) bounded from below by  $\varphi(y) > 0$ , then applying the first part of Theorem 5.2 we have

$$\frac{\varphi(x)}{e^{\rho x}} \geq \frac{1}{\sigma(S)} \frac{\varphi(y)}{e^{\rho y}}. \quad \square$$

Before passing to solutions which are not integrable we will prove the following lemma.

**Lemma 5.5.** *If  $a$  is a real number,  $\varphi: (a, +\infty) \rightarrow \mathbb{R}$  is Lebesgue integrable on every finite interval contained in  $(a, +\infty)$ , and  $\lambda > 0$ , then*

$$\limsup_{x \rightarrow +\infty} \frac{\varphi(x)}{e^{\lambda x}} \geq \lambda \limsup_{x \rightarrow +\infty} \frac{\int_a^x \varphi(y) dy}{e^{\lambda x}}.$$

*Proof.* Put

$$d = \limsup_{x \rightarrow +\infty} \frac{\int_a^x \varphi(y) dy}{e^{\lambda x}},$$

suppose

$$\limsup_{x \rightarrow +\infty} \frac{\varphi(x)}{e^{\lambda x}} < \lambda d$$

and choose real numbers  $D < \lambda d$  and  $b \geq a$  so that

$$\varphi(x) \leq D e^{\lambda x} \quad \text{for } x > b.$$

Then

$$\int_b^x \varphi(y) dy \leq D \int_b^x e^{\lambda y} dy = \frac{D}{\lambda} (e^{\lambda x} - e^{\lambda b}),$$

i.e.,

$$\frac{\int_b^x \varphi(y) dy}{e^{\lambda x}} \leq \frac{D}{\lambda} (1 - e^{\lambda(b-x)}) \quad \text{for } x > b.$$

Consequently

$$d = \limsup_{x \rightarrow +\infty} \frac{\int_b^x \varphi(y) dy}{e^{\lambda x}} \leq \frac{D}{\lambda} < d;$$

a contradiction. □

**Corollary 5.6.** *If  $\sigma(S) > 1$  and  $\varphi: (a, +\infty) \rightarrow \mathbb{R}$  is a positive regular solution of (1.3) which is not integrable on a half-line infinite from the right, then*

$$0 < \frac{\rho}{\sigma(S)} \limsup_{x \rightarrow +\infty} \frac{\int_c^x \varphi(y) dy}{e^{\rho x}} \leq \limsup_{x \rightarrow +\infty} \frac{\varphi(x)}{e^{\rho x}} \quad (5.3)$$

for any real number  $c \geq a$  and

$$\int_c^{+\infty} \frac{\varphi(x)}{e^{\rho x}} dx = +\infty \quad \text{for } c \geq a; \quad (5.4)$$

moreover, either  $\rho$  is a root of (1.2) or

$$\limsup_{x \rightarrow +\infty} \frac{\varphi(x)}{e^{\rho x}} = +\infty. \quad (5.5)$$

*Proof.* We can assume that  $c = a \in \mathbb{R}$ . Let  $\psi$  denote the function (4.9). Making use of [13, Lemma 3] it is easy to check that  $\psi$  is a solution of (1.3). Obviously  $\psi$  is increasing and has infinite limit at  $+\infty$ . In particular it is positive on an interval of the form  $(b, +\infty)$ . Hence, applying Corollary 5.4, we have

$$0 < \frac{1}{\sigma(S)} \sup \left\{ \frac{\psi(x)}{e^{\rho x}} : x \in (b, +\infty) \right\} \leq \liminf_{x \rightarrow +\infty} \frac{\psi(x)}{e^{\rho x}},$$

which together with the definition of  $\psi$  implies that

$$0 < \frac{1}{\sigma(S)} \limsup_{x \rightarrow +\infty} \frac{\int_a^x \varphi(y) dy}{e^{\rho x}} \leq \liminf_{x \rightarrow +\infty} \frac{\int_a^x \varphi(y) dy}{e^{\rho x}}.$$

This jointly with Lemma 5.5 gives (5.3).

If  $\rho$  is a root of (1.2), then (due to Remark 5.3)  $M = -m$  a.e. and applying Theorem 4.2 we obtain (5.4).

Assume now that  $\rho$  is not a root of (1.2). According to Theorem 5.2 we have then

$$\lim_{x \rightarrow +\infty} \frac{\psi(x)}{e^{\rho x}} = +\infty$$

and, consequently,

$$\lim_{x \rightarrow +\infty} \frac{\int_a^x \varphi(y) dy}{e^{\rho x}} = +\infty.$$

Making use of Lemma 5.5 once more we get (5.5), and noting that

$$\frac{\varphi(y)}{e^{\rho x}} \leq \frac{\varphi(y)}{e^{\rho y}} \quad \text{for } x \geq y > a,$$

also (5.4). □

It turns out that in Theorem 5.2 as well as in Corollaries 5.4, 5.6 we cannot replace inequality (1.3) by (1.4), and if  $\sigma(S) \leq 1$ , then inequality (1.3) can have a solution tending to infinity more slowly than any exponential function, which show the next two remarks.

**Remark 5.7.** If either  $\sigma(S) > 1$ , or  $\sigma(S) = 1$  and  $\int_S M(s)\sigma(ds) \geq 0$ , then the identity is an increasing solution of (1.4) on some interval of the form  $(a, +\infty)$  and

$$\lim_{x \rightarrow +\infty} \frac{x}{e^{\lambda x}} = 0 \quad \text{for } \lambda > 0.$$

**Remark 5.8.** If either  $\sigma(S) < 1$ , or  $\sigma(S) = 1$  and  $\int_S M(s)\sigma(ds) \leq 0$ , then the identity is a solution of (1.3) on some interval of the form  $(a, +\infty)$ .

**Theorem 5.9.** *If  $\sigma(S) < 1$  and  $\varphi : (a, +\infty) \cap W \rightarrow (0, +\infty)$  is an increasing solution of (1.4), then*

$$\frac{\varphi(x)}{e^{-\rho x}} \geq \sigma(S) \sup \left\{ \frac{\varphi(y)}{e^{-\rho y}} : y \in (a, x] \cap W \right\} \quad \text{for } x \in (a, +\infty) \cap W; \quad (5.6)$$

moreover, either  $-\rho$  is a root of (1.2) or

$$\lim_{x \rightarrow +\infty} \frac{\varphi(x)}{e^{-\rho x}} = +\infty.$$

*Proof.* We show at first by induction that

$$\varphi(y) \leq \sigma(S)^n \varphi(x) \quad \text{for } y \in (a, +\infty) \cap W, \quad x \in (y + nm, +\infty) \cap W \quad (5.7)$$

and  $n \in \mathbb{N} \cup \{0\}$ .

To do this fix  $n \in \mathbb{N}$ ,  $y \in (a, +\infty) \cap W$  and  $x \in (y + (n + 1)m, +\infty) \cap W$ . If  $m \in M(S)$ , then  $x - m \in W$ . If  $m \notin M(S)$ , then the group  $\langle M(S) \rangle$  contains arbitrarily small elements and thus it is dense. Consequently the invariant set  $W$  is also dense. In both cases the set  $(y + nm, x - m] \cap W$  is nonempty. Taking a point  $z$  from this set and making use of the inductive assumption as well as the monotonicity of  $\varphi$  we have

$$\varphi(y) \leq \sigma(S)^n \varphi(z) \leq \sigma(S)^n \int_S \varphi(z + M(s)) \sigma(ds) \leq \sigma(S)^{n+1} \varphi(x).$$

Passing to the proof of (5.6) fix  $x \in (a, +\infty) \cap W$  and  $y \in (a, x] \cap W$ . We will show to complete the proof of (5.6), that

$$\frac{\varphi(x)}{e^{-\rho x}} \geq \sigma(S) \frac{\varphi(y)}{e^{-\rho y}}. \quad (5.8)$$

Since  $\sigma(S) < 1$ , we can assume that  $y < x$ . Let

$$n := \max\{k \in \mathbb{N} \cup \{0\} : x > y + km\}.$$

Then

$$\frac{\sigma(S)^{-n}}{e^{-\rho x}} = e^{(\frac{x}{m} - n) \ln \sigma(S)} \geq e^{(\frac{y}{m} + 1) \ln \sigma(S)} = \frac{\sigma(S)}{e^{-\rho y}}$$

and taking into account also (5.7) we get

$$\frac{\varphi(x)}{e^{-\rho x}} \geq \frac{\sigma(S)^{-n} \varphi(y)}{e^{-\rho x}} \geq \sigma(S) \frac{\varphi(y)}{e^{-\rho y}},$$

which ends the proof of (5.8).

Assume now that

$$\liminf_{x \rightarrow +\infty} \frac{\varphi(x)}{e^{-\rho x}} < +\infty.$$

Since, due to (5.6), we have

$$\liminf_{x \rightarrow +\infty} \frac{\varphi(x)}{e^{-\rho x}} \geq \sigma(S) \sup \left\{ \frac{\varphi(y)}{e^{-\rho y}} : y \in (a, +\infty) \cap W \right\},$$

then

$$\begin{aligned}
 0 &< \sigma(S) \sup \left\{ \frac{\varphi(x)}{e^{-\rho x}} : x \in (a, +\infty) \cap W \right\} \leq \liminf_{x \rightarrow +\infty} \frac{\varphi(x)}{e^{-\rho x}} \leq \\
 &\leq \limsup_{x \rightarrow +\infty} \frac{\varphi(x)}{e^{-\rho x}} \leq \frac{1}{\sigma(S)} \liminf_{x \rightarrow +\infty} \frac{\varphi(x)}{e^{-\rho x}} < +\infty
 \end{aligned}$$

and it follows from [14, Theorem 2(ii)] that  $u(-\rho) \geq 1$ . It is enough now to observe that

$$u(-\rho) = \int_S e^{-\rho M(s)} \sigma(ds) \leq \int_S e^{-\rho m} \sigma(ds) = 1. \quad \square$$

From Theorem 5.9 we draw (similarly as we drew Corollary 5.6 from Theorem 5.2) the following corollary concerning solutions which are not integrable on a half-line infinite from the right.

**Corollary 5.10.** *If  $\sigma(S) < 1$  and  $\varphi: (a, +\infty) \rightarrow \mathbb{R}$  is a positive regular solution of (1.4) which is not integrable on a half-line infinite from the right, then*

$$0 < -\rho \sigma(S) \limsup_{x \rightarrow +\infty} \frac{\int_c^x \varphi(y) dy}{e^{-\rho x}} \leq \limsup_{x \rightarrow +\infty} \frac{\varphi(x)}{e^{-\rho x}}$$

for any real number  $c \geq a$  and

$$\int_c^{+\infty} \frac{\varphi(x)}{e^{-\rho x}} dx = +\infty \quad \text{for } c \geq a;$$

moreover, either  $-\rho$  is a root of (1.2) or

$$\limsup_{x \rightarrow +\infty} \frac{\varphi(x)}{e^{-\rho x}} = +\infty.$$

Remarks 5.8 and 5.7 show that in Theorem 5.9 as well as in Corollary 5.10 we cannot replace inequality (1.4) by (1.3), and if  $\sigma(S) \geq 1$ , then (1.4) can have a solution tending to infinity more slowly than any exponential function.

As follows from Corollaries 5.6 and 5.10, if  $\sigma(S) \neq 1$ , then any positive regular solution of (1.1) which is not integrable on a half-line infinite from the right tends to infinity on some sequence faster than some exponential function. Simultaneously, according to [13, Theorems 16 and 19] and Theorem 4.6, no such a solution can tend to infinity faster than  $e^{\lambda x}$  with  $\lambda > \lambda_2 \geq 0$ . Note that similar statements are not true in the case of inequalities – cf. Remarks 5.7, 5.8 and Theorems 4.15, 4.16.

## 6. ASYMPTOTIC BEHAVIOUR OF DECREASING SOLUTIONS

**Theorem 6.1.** *If  $\varphi: (a, +\infty) \cap W \rightarrow [0, +\infty)$  is a decreasing solution of (1.1), then*

$$\varphi(x) e^{|\rho|x} \leq \max \left\{ \sigma(S), \frac{1}{\sigma(S)} \right\} \cdot \inf \{ \varphi(y) e^{|\rho|y} : y \in (a, x] \cap W \} \quad \text{for } x \in (a, +\infty) \cap W,$$

and, moreover, either  $-\rho$  is a root of (1.2) or

$$\lim_{x \rightarrow +\infty} \varphi(x)e^{|\rho|x} = 0.$$

This result is an immediate consequence of theorems concerning inequalities given below.

**Theorem 6.2.** *If  $\sigma(S) > 1$  and  $\varphi: (a, +\infty) \cap W \rightarrow [0, +\infty)$  is a decreasing solution of (1.3), then*

$$\varphi(x)e^{\rho x} \leq \sigma(S) \inf\{\varphi(y)e^{\rho y} : y \in (a, x] \cap W\} \quad \text{for } x \in (a, +\infty) \cap W;$$

moreover, either  $-\rho$  is a root of (1.2) or

$$\lim_{x \rightarrow +\infty} \varphi(x)e^{\rho x} = 0.$$

*Proof.* We argue as in the proof of Theorem 5.9 using now [14, Theorem 2(i)]. □

From Theorem 6.2 we draw a corollary concerning integrable solutions. In the proof we will need the following lemma. We omit its proof (being similar to that of Lemma 5.5).

**Lemma 6.3.** *If  $\varphi: (a, +\infty) \rightarrow \mathbb{R}$  is a Lebesgue integrable function and  $\lambda > 0$ , then*

$$\liminf_{x \rightarrow +\infty} \varphi(x)e^{\lambda x} \leq \lambda \liminf_{x \rightarrow +\infty} e^{\lambda x} \int_x^{+\infty} \varphi(y)dy.$$

**Corollary 6.4.** *If  $\sigma(S) > 1$  and  $\varphi: (a, +\infty) \rightarrow \mathbb{R}$  is a positive regular solution of (1.3) integrable on a half-line infinite from the right, then*

$$\liminf_{x \rightarrow +\infty} \varphi(x)e^{\rho x} \leq \rho \sigma(S) \inf \left\{ e^{\rho x} \int_x^{+\infty} \varphi(y)dy : x \in (a, +\infty) \right\}; \tag{6.1}$$

moreover, either  $-\rho$  is a root of (1.2) or

$$\liminf_{x \rightarrow +\infty} \varphi(x)e^{\rho x} = 0. \tag{6.2}$$

*Proof.* According to [13, Lemma 4] the function (4.4) is a decreasing solution of (1.3) and using Theorem 6.2 we get

$$\limsup_{x \rightarrow +\infty} e^{\rho x} \int_x^{+\infty} \varphi(y)dy \leq \sigma(S) \inf \left\{ e^{\rho x} \int_x^{+\infty} \varphi(y)dy : x \in (a, +\infty) \right\},$$



which, due to Lemma 6.3, implies (6.1).

If  $-\rho$  is not a root of (1.2), then by virtue of Theorem 6.2 we have

$$\lim_{x \rightarrow +\infty} e^{\rho x} \int_x^{+\infty} \varphi(y) dy = 0$$

and making use of Lemma 6.3 once more we obtain (6.2). □

The next two remarks show that in Theorem 6.2 as well as in Corollary 6.4 we cannot replace inequality (1.3) by (1.4), and if  $\sigma(S) < 1$  (or  $\sigma(S) = 1$  and  $\int_S M(s)\sigma(ds) > 0$ ), then for any exponential function there is a decreasing solution of (1.3) vanishing at infinity more slowly than that function.

**Remark 6.5.** If either  $\sigma(S) > 1$ , or  $\sigma(S) = 1$  and  $\int_S M(s)\sigma(ds) < 0$ , then  $e^{\lambda x}$  is a solution of (1.4) for any  $\lambda$  from some left-hand-side vicinity of zero; moreover, if  $M \leq 0$  a.e. and  $\sigma(S) \geq 1$ , then the function  $\frac{1}{x^2}$ ,  $x > m$ , is a decreasing integrable solution of (1.4) and

$$\lim_{x \rightarrow +\infty} \frac{1}{x^2} e^{\lambda x} = +\infty \quad \text{for } \lambda > 0.$$

**Remark 6.6.** If either  $\sigma(S) < 1$ , or  $\sigma(S) = 1$  and  $\int_S M(s)\sigma(ds) > 0$ , then  $e^{\lambda x}$  is a solution of (1.3) for any  $\lambda$  from some left-hand-side vicinity of zero; moreover, if  $M \geq 0$  a.e. and  $\sigma(S) \leq 1$ , then the function  $\frac{1}{x^2}$ ,  $x > m$ , is a decreasing integrable solution of (1.3) which vanishes at infinity more slowly than any exponential function.

**Theorem 6.7.** *If  $\sigma(S) < 1$  and  $\varphi: (a, +\infty) \cap W \rightarrow [0, +\infty)$  is a nonzero solution of (1.4) bounded from above by a constant  $C$ , then  $a$  is a real number and*

$$\varphi(x)e^{-\rho x} \leq \frac{C}{\sigma(S)} e^{-\rho a} \quad \text{for } x \in (a, +\infty) \cap W;$$

moreover, either  $\rho$  is a root of (1.2) or

$$\lim_{x \rightarrow +\infty} \varphi(x)e^{-\rho x} = 0.$$

*Proof.* We proceed as in the proof of Theorem 5.2 applying now [14, Theorem 2(ii)]. □

From Theorem 6.7 we can easily draw the following two corollaries.

**Corollary 6.8.** *If  $\sigma(S) < 1$  and  $\varphi: (a, +\infty) \cap W \rightarrow [0, +\infty)$  is a decreasing solution of (1.4), then*

$$\varphi(x)e^{-\rho x} \leq \frac{1}{\sigma(S)} \inf\{\varphi(y)e^{-\rho y} : y \in (a, x] \cap W\} \quad \text{for } x \in (a, +\infty) \cap W.$$

**Corollary 6.9.** *If  $\sigma(S) < 1$  and  $\varphi: (a, +\infty) \rightarrow \mathbb{R}$  is a positive regular solution of (1.4) integrable on a half-line infinite from the right, then*

$$\liminf_{x \rightarrow +\infty} \varphi(x)e^{-\rho x} \leq -\frac{\rho}{\sigma(S)} \inf \left\{ e^{-\rho x} \int_x^{+\infty} \varphi(y)dy : x \in (a, +\infty) \right\};$$

moreover, either  $\rho$  is a root of (1.2) or

$$\liminf_{x \rightarrow +\infty} \varphi(x)e^{-\rho x} = 0.$$

Remarks 6.6, 6.5 show that in Theorem 6.7 as well as in Corollaries 6.8, 6.9 we cannot replace inequality (1.4) by (1.3), and if  $\sigma(S) > 1$  (or  $\sigma(S) = 1$  and  $\int_S M(s)\sigma(ds) < 0$ ), then for any exponential function there is a decreasing solution of (1.4) vanishing at infinity more slowly than that function.

Using [13, Theorem 34] and Corollaries 6.4, 6.9 we obtain our last result.

**Corollary 6.10.** *If  $\varphi: (a, +\infty) \rightarrow \mathbb{R}$  is a positive regular solution of (1.1) vanishing at  $+\infty$ , then*

$$\begin{aligned} \liminf_{x \rightarrow +\infty} \varphi(x)e^{|\rho|x} &\leq \\ &\leq |\rho| \max \left\{ \sigma(S), \frac{1}{\sigma(S)} \right\} \cdot \inf \left\{ e^{|\rho|x} \int_x^{+\infty} \varphi(y)dy : x \in (a, +\infty) \right\} < +\infty, \end{aligned}$$

and, moreover, either  $-|\rho|$  is a root of (1.2) or

$$\liminf_{x \rightarrow +\infty} \varphi(x)e^{|\rho|x} = 0.$$

As follows from Corollaries 6.4, 6.9 and 6.10, if  $\sigma(S) \neq 1$ , then any positive regular solution of (1.1) integrable on a half-line infinite from the right or vanishing at  $+\infty$  tends to zero on some sequence faster than some exponential function. Simultaneously, according to [13, Theorems 27, 29, Corollary 36] and Theorem 4.1, no such solution can tend to zero faster than  $e^{\lambda_1 x}$ . Note that similar statements are not true in the case of inequalities – cf. Remarks 6.5, 6.6 and Theorems 4.12, 4.13.

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