EXISTENCE OF PERIODIC SOLUTIONS FOR TOTALLY NONLINEAR NEUTRAL DIFFERENTIAL EQUATIONS WITH FUNCTIONAL DELAY

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Abstract. We use a variant of Krasnoselskii's fixed point theorem by T.A. Burton to show that the nonlinear neutral differential equation with functional delay

$$x'(t) = -a(t)h(x(t)) + c(t)x'(t - g(t)) + q\big(t, x(t), x(t - g(t))\big)$$

has a periodic solution.

Keywords: fixed point, large contraction, periodic solution, totally nonlinear neutral equation.

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1. INTRODUCTION

In this work, we consider the totally nonlinear neutral differential equation

$$x'(t) = -a(t)h(x(t)) + c(t)x'(t - g(t)) + q\big(t, x(t), x(t - g(t))\big). \tag{1.1}$$

Equations of the form similar to equation (1.1), where $h(x(t)) = x(t)$, has gained the attention of many researchers in recent times, see [2, 6–8, 10–12] and the references therein. Here $a(t)$ is a real valued function, $c(t)$ is continuously differentiable, $g(t)$ is twice continuously differentiable, $h : \mathbb{R} \to \mathbb{R}$ is continuous with respect to its argument and $q : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is also continuous with respect to its arguments. Equation (1.1) is such that the variation of parameters cannot be applied directly. We therefore resort to the idea of adding and subtracting a linear term. As noted by Burton in [1], the added terms destroy a contraction already present in part of the equation but replaces it with the so called large contraction mapping which is suitable for fixed point theory. Thus, in this paper we use the concept of the large contraction to study the existence of periodic solutions of (1.1).

2. PRELIMINARIES

Let $T > 0$ and define the set $P_T = \{\phi \in C(\mathbb{R}, \mathbb{R}) : \phi(t+T) = \phi(t)\}$ and the norm $\|x(t)\| = \max_{t\in[0,T]} |x(t)|$, where C is the space of continuous real valued functions. Then $(P_T, \|.\|)$ is a Banach space. Also, for any $L > 0$, define

$$\mathbb{M}_L = \{\varphi \in P_T \ : \ \|\varphi\| \leq L\}, \tag{2.1}$$

In this paper we make the following assumptions.

$$a(t+T) = a(t), \quad c(t+T) = c(t), \quad g(t+T) = g(t), \quad g(t) \geq g^* > 0 \tag{2.2}$$

with $c(t)$ continuously differentiable, $g(t)$ twice continuously differentiable and g^* is constant. Also,

$$\int_0^T a(s)ds > 0. \tag{2.3}$$

We also assume that $q(t, x, y)$ is continuous and periodic in t and Lipschitz continuous in x and y, that is

$$q(t+T, x, y) = q(t, x, y) \tag{2.4}$$

and for some positive constants K and E,

$$|q(t, x, y) - q(t, z, w)| \leq K\|x - z\| + E\|y - w\|. \tag{2.5}$$

Also, we assume that for all t, $0 \leq t \leq T$,

$$g'(t) \neq 1. \tag{2.6}$$

Since $g(t)$ is periodic, condition (2.6) implies that $g'(t) < 1$.

Lemma 2.1. *Suppose* (2.2)–(2.3) *and* (2.6) *hold. If* $x(t) \in P_T$, *then* $x(t)$ *is a solution of equation* (1.1) *if and only if*

$$x(t) = \frac{c(t)}{1-g'(t)} x(t-g(t)) + \left(1 - e^{-\int_{t-T}^{t} a(s)ds}\right)^{-1} \times$$
$$\times \int_{t-T}^{t} \big[a(u)H(x(u)) - r(u)x(u-g(u)) + q(u, x(u), x(u-g(u)))\big] e^{-\int_u^t a(s)ds} du, \tag{2.7}$$

where

$$r(u) = \frac{\Big(c'(u) - c(u)a(u)\Big)\Big(1 - g'(u)\Big) + g''(u)c(u)}{(1-g'(u))^2}, \tag{2.8}$$

and

$$H(x(t)) = x(t) - h(x(t)). \tag{2.9}$$

Proof. Let $x(t) \in P_T$ be a solution of (1.1). We first rewrite (1.1) in the form

$$x'(t) + a(t)x(t) = a(t)H(x(t)) + c(t)x'(t-g(t)) + q\big(t, x(t), x(t-g(t))\big). \tag{2.10}$$

Multiply both sides of (2.10) by $e^{\int_0^t a(s)ds}$ and then integrate from $t-T$ to t to obtain

$$\int_{t-T}^{t} \left[x(u)e^{\int_0^u a(s)ds}\right]' du =$$
$$= \int_{t-T}^{t} \big[a(u)H(x(u)) + c(u)x'(u-g(u)) + q(u, x(u), x(u-g(u)))\big] e^{\int_0^u a(s)ds} du.$$

Thus we obtain,

$$x(t)e^{\int_0^t a(s)ds} - x(t-T)e^{\int_0^{t-T} a(s)ds} =$$
$$= \int_{t-T}^{t} \big[a(u)H(x(u)) + c(u)x'(u-g(u)) + q(u, x(u), x(u-g(u)))\big] e^{\int_0^u a(s)ds} du.$$

By dividing both sides of the above equation by $\exp(\int_0^t a(s)ds)$ and using the fact that $x(t) = x(t-T)$, we obtain

$$x(t) = \left(1 - e^{-\int_{t-T}^{t} a(s)ds}\right)^{-1} \times$$
$$\times \int_{t-T}^{t} \big[a(u)H(x(u)) + c(u)x'(u-g(u)) + q(u, x(u), x(u-g(u)))\big] e^{-\int_u^t a(s)ds} du. \tag{2.11}$$

Rewrite

$$\int_{t-T}^{t} c(u)x'(u-g(u))e^{-\int_u^t a(s)ds} du =$$
$$= \int_{t-T}^{t} \frac{c(u)x'(u-g(u))(1-g'(u))}{(1-g'(u))} e^{-\int_u^t a(s)ds} du.$$

Using integration by parts on the above integral with

$$U = \frac{c(u)}{1-g'(u)} e^{-\int_u^t a(s)ds} \quad \text{and} \quad dV = x'(u-g(u))(1-g'(u))du$$

we obtain

$$\int_{t-T}^{t} c(u)x'(u-g(u))e^{-\int_u^t a(s)ds} du = \tag{2.12}$$

$$= \frac{c(t)}{1-g'(t)} x(t-g(t)) \Big(1 - e^{-\int_{t-T}^{t} a(s)ds} \Big) - \int_{t-T}^{t} r(u) e^{-\int_u^t a(s)ds} x(u-g(u))du,$$

where $r(u)$ is given by (2.8). Then substituting (2.12) into (2.11) gives the desired results. Since each step in the above work is reversible, the proof is complete. □

In the analysis, we employ a fixed point theorem in which the notion of a large contraction is required as one of the sufficient conditions. First, we give the following definition which can be found in [4].

Definition 2.2. Let $(\mathbb{M}, d)$ be a metric space and $B : \mathbb{M} \to \mathbb{M}$. B is said to be a large contraction if $\psi, \varphi \in \mathbb{M}$, with $\psi \neq \varphi$ then $d(B\varphi, B\psi) < d(\varphi, \psi)$ and if for all $\epsilon > 0$ there exists $\delta < 1$ such that

$$[\psi, \varphi \in \mathbb{M}, d(\varphi, \psi) \geq \epsilon] \Rightarrow d(B\varphi, B\psi) \leq \delta d(\varphi, \psi).$$

Theorem 2.3. *Let $(\mathbb{M}, d)$ be a complete metric space and B a large contraction. Suppose there is an $x \in \mathbb{M}$ and an $L > 0$, such that $d(x, B^n x) \leq L$ for all $n \geq 1$. Then B has a unique fixed point in $\mathbb{M}$.*

The next theorem, which constitutes a basis for our main result, is a reformulated version of Krasnoselskii's fixed point theorem.

Theorem 2.4 ([4]). *Let $\mathbb{M}$ be a bounded convex non-empty subset of a Banach space $(S, \|.\|)$. Suppose that A, B map $\mathbb{M}$ into $\mathbb{M}$ and that:*

(i) *for all $x, y \in \mathbb{M} \Rightarrow Ax + By \in \mathbb{M}$,*
(ii) *A is continuous and AM is contained in a compact subset of M,*
(iii) *B is a large contraction.*

Then there is a $z \in \mathbb{M}$ with $z = Az + Bz$.

3. EXISTENCE OF PERIODIC SOLUTION

In this section we state and prove our main results. In view of this we first define the operator P by

$$(P\varphi)(t) = \frac{c(t)}{1-g'(t)}\varphi(t-g(t)) + \left(1 - e^{-\int_{t-T}^{t} a(s)ds}\right)^{-1} \times \\ \times \int_{t-T}^{t} \big[a(u)H(\varphi(u)) - r(u)\varphi(u-g(u)) + \\ + q(u,\varphi(u),\varphi(u-g(u)))\big] e^{-\int_{u}^{t} a(s)ds} du, \tag{3.1}$$

where r and H are given in (2.8) and (2.9), respectively. It therefore follows from Lemma 2.1 that fixed points of P are solutions of (1.1) and *vice versa*.
In order to employ Theorem 2.4 we need to express the operator P as a sum of two operators, one of which is completely continuous and the other a large contraction. Let $(P\varphi)(t) = A\varphi(t) + B\varphi(t)$, where $A, B : P_T \to P_T$ are defined by

$$(B\varphi)(t) = \left(1 - e^{-\int_{t-T}^{t} a(s)ds}\right)^{-1} \times \int_{t-T}^{t} \big[a(u)H(\varphi(u))\big] e^{-\int_{u}^{t} a(r)dr} du, \tag{3.2}$$

and

$$(A\varphi)(t) = \frac{c(t)}{1-g'(t)}\varphi(t-g(t)) + \left(1 - e^{-\int_{t-T}^{t} a(s)ds}\right)^{-1} \times \\ \times \int_{t-T}^{t} \big[-r(u)\varphi(u-g(u)) + q(u,\varphi(u),\varphi(u-g(u)))\big] e^{-\int_{u}^{t} a(s)ds} du. \tag{3.3}$$

In the rest of the paper we require the following conditions.

$$KL + EL + |q(t,0,0)| \leq \beta L a(t), \tag{3.4}$$

$$|r(t)| \leq \delta a(t), \tag{3.5}$$

$$\max_{t\in[0,T]} \left| \frac{c(t)}{(1-g'(t))} \right| = \alpha, \tag{3.6}$$

and

$$J(\beta + \alpha + \delta) \leq 1, \tag{3.7}$$

where $\alpha,\ \beta,\ \delta, L$ and J are constants with $J \geq 3$.

Next we state our main result and present its proof in four lemmas.

Theorem 3.1. *Let L be a fixed positive number and let $(P_T, \|.\|)$ be the Banach space of continuous T-periodic real functions. Suppose* (2.2)–(2.4), *and* (3.4)–(3.7) *hold. Then equation* (1.1) *possesses a periodic solution in the subset $\mathbb{M}_L$.*

The proof is based on the following four lemmas.

Lemma 3.2. *Suppose that conditions* (2.2)–(2.4) *and* (3.4)–(3.7) *hold. Then for L defined in Theorem 3.1, $A : \mathbb{M}_L \to \mathbb{M}_L$ is continuous in the supremum norm and maps $\mathbb{M}_L$ into a compact subset of $\mathbb{M}_L$.*

Proof. A change of variable in (3.3) shows that $(A\varphi)(t+T) = (A\varphi)(t)$. Note that

$$|q(t,x,y)| \le |q(t,x,y) - q(t,0,0)| + |q(t,0,0)| \le K|x| + E|y| + |q(t,0,0)|.$$

We will first show that A maps $\mathbb{M}_L$ into itself. Thus, for any $\varphi \in \mathbb{M}_L$, we have

$$
\begin{aligned}
|(A\varphi)(t)| \le & \left|\frac{c(t)\varphi(t-g(t))}{1-g'(t)}\right| + \left(1 - e^{-\int_{t-T}^{t} a(s)ds}\right)^{-1} \times \\
& \times \int_{t-T}^{t} |r(u)\varphi(u-g(u))| e^{-\int_{u}^{t} a(s)ds} du + \\
& + \left(1 - e^{-\int_{t-T}^{t} a(s)ds}\right)^{-1} \times \int_{t-T}^{t} |q(u,\varphi(u),\varphi(u-g(u)))| e^{-\int_{u}^{t} a(s)ds} du \le \\
\le\; & \alpha L + \left(1 - e^{-\int_{t-T}^{t} a(s)ds}\right)^{-1} \times \int_{t-T}^{t} \delta a(u) L e^{-\int_{u}^{t} a(s)ds} du + \\
& + \left(1 - e^{-\int_{t-T}^{t} a(s)ds}\right)^{-1} \times \int_{t-T}^{t} (KL + EL + |q(t,0,0)|) e^{-\int_{u}^{t} a(s)ds} du \le \\
\le\; & \alpha L + \delta L \left(1 - e^{-\int_{t-T}^{t} a(s)ds}\right)^{-1} \times \int_{t-T}^{t} a(u) e^{-\int_{u}^{t} a(s)ds} du + \\
& + \beta L \left(1 - e^{-\int_{t-T}^{t} a(s)ds}\right)^{-1} \times \int_{t-T}^{t} a(u) e^{-\int_{u}^{t} a(s)ds} du \le \\
\le\; & (\alpha + \delta + \beta) L \le \frac{L}{J} < L.
\end{aligned}
$$

Thus showing that A maps $\mathbb{M}_L$ into itself.

We next show that A is continuous. Let $\varphi, \psi \in \mathbb{M}_L$, and let

$$\begin{aligned} a &= \max_{t\in[0,T]} \left(1 - e^{-\int_{t-T}^{t} a(s)ds}\right)^{-1}, & b &= \max_{u\in[t-T,t]} e^{-\int_{u}^{t} a(s)ds}, \\ \sigma &= \max_{t\in[0,T]} r(t), & \lambda &= \max_{t\in[0,T]} |q(t,0,0)|, \\ \nu &= \max_{t\in[0,T]} \left|\frac{c'(t)}{(1-g'(t))}\right|, & \mu &= \max_{t\in[0,T]} \left|\frac{g''(t)c(t)}{(1-g'(t))^2}\right|. \end{aligned} \tag{3.8}$$

Given $\epsilon > 0$, take $\delta = \epsilon/F$ such that $\|\varphi - \psi\| < \delta$. Then we get

$$\begin{aligned} \big\|\big(A\varphi\big)(t) - \big(A\psi\big)(t)\big\| &\le \alpha\|\varphi-\psi\| + ab\int_{t-T}^{t} \big[L\|\varphi-\psi\| + E\|\varphi-\psi\| + \sigma\|\varphi-\psi\|\big]du \le \\ &\le F\|\varphi-\psi\| < \epsilon \end{aligned}$$

where $F = \alpha + Tab[\sigma + L + E]$. This proves A is continuous. To show A is compact, we let $\varphi_n \in \mathbb{M}_L$ where n is a positive integer. Then as before we have that

$$\|A(\varphi_n(t))\| \le L. \tag{3.9}$$

Moreover, a direct calculation shows that

$$\begin{aligned} (A\varphi_n)'(t) &= q(t,\varphi_n(t),\varphi_n(t-g(t))) - r(t)\varphi_n(t-g(t)) - a(t)\left(1 - e^{-\int_{t-T}^{t} a(s)ds}\right)^{-1} \times \\ &\times \int_{t-T}^{t} [q(u,\varphi_n(u),\varphi_n(u-g(u))) - r(t)\varphi_n(u-g(u))]e^{-\int_{u}^{t} a(s)ds} du + \\ &+ \frac{c'(t)\varphi_n(t) + c(t)\varphi_n'(t)}{1-g'(t)} + \frac{g''(t)c(t)\varphi_n(t)}{(1-g'(t))^2}. \end{aligned}$$

By invoking conditions (2.5), (3.4)–(3.6), (3.8) and (3.9) we obtain

$$|(A\varphi_n)'(t)| \le KL + EL + \lambda + \delta\|a\|L + \|a\|L + \nu L + \alpha L' + \mu L \le D,$$

for some positive constant D. Hence the sequence $(A\varphi_n)$ is uniformly bounded and equicontinuous. The Ascoli-Arzela theorem implies that the subsequence $(A\varphi_{n_k})$ of $(A\varphi_n)$ converges uniformly to a continuous T-periodic function. Thus, A is compact. □

Lemma 3.3. *Suppose* (2.2)–(2.5), *and* (3.4) *hold. Suppose also that for L defined in Theorem 3.1,*

$$\left(1 - e^{-\int_{t-T}^{t} a(s)ds}\right)^{-1} \times \int_{t-T}^{t} \big[|a(u)||H(\varphi(u))|\big]e^{-\int_{u}^{t} a(r)dr} du \le \frac{(J-1)L}{J}. \tag{3.10}$$

For B, A defined by (3.2) *and* (3.3), *if $\varphi, \psi \in \mathbb{M}_L$ are arbitrary, then*

$$A\varphi + B\psi : \mathbb{M}_L \to \mathbb{M}_L.$$

Proof. Let $\varphi, \psi \in \mathbb{M}_L$ be arbitrary. Using the definition of B and the result of Lemma 3.2, we obtain

$$\begin{aligned}
&|(A\varphi)(t) + (B\psi)(t)| \leq \\
&\leq |\frac{c(t)}{1-g'(t)}\varphi(t-g(t))| + \left(1 - e^{-\int\limits_{t-T}^{t} a(s)ds}\right)^{-1} \times \\
&\quad \times \int\limits_{t-T}^{t} \Big[|r(u)\varphi(u-g(u))| + |q(u,\varphi(u),\varphi(u-g(u)))|\Big] e^{-\int\limits_{u}^{t} a(s)ds} du + \\
&\quad + \left(1 - e^{-\int\limits_{t-T}^{t} a(s)ds}\right)^{-1} \times \int\limits_{t-T}^{t} \big|a(u)H(\varphi(u))\big| e^{-\int\limits_{u}^{t} a(r)dr} du \leq \\
&\leq \frac{L}{J} + \frac{(J-1)L}{J} = L.
\end{aligned}$$

Thus $A\varphi + B\psi \in \mathbb{M}_L$. This completes the proof. $\square$

In the next lemma we prove that H is a large contraction on $\mathbb{M}_L$. To this end we make the following assumptions on the function $h : \mathbb{R} \to \mathbb{R}$.

(H1) h is continuous and differentiable on $U_L = [-L, L]$.
(H2) h is strictly increasing on U_L.
(H3) $\sup_{s\in U_L} h'(s) \leq 1$.
(H4) $(s-r)\Big\{\sup_{t\in U_L} h'(t)\Big\} \geq h(s) - h(r) \geq (s-r)\Big\{\inf_{t\in U_L} h'(t)\Big\} \geq 0$ for $s, r \in U_L$ with $s \geq r$.

Lemma 3.4. *Let $h : \mathbb{R} \to \mathbb{R}$ be a function satisfying $(H1)-(H4)$. Then for L defined in Theorem 3.1, the mapping H is a large contraction on the set $\mathbb{M}_L$.*

Proof. Let $\phi, \varphi \in \mathbb{M}_L$ with $\phi \neq \varphi$. Then $\phi(t) \neq \varphi(t)$ for some $t \in \mathbb{R}$. Define the set

$$D(\phi, \varphi) = \Big\{\ t \in \mathbb{R} : \phi(t) \neq \varphi(t)\Big\}.$$

Note that $\varphi(t) \in U_L$ for all $t \in \mathbb{R}$ whenever $\varphi \in \mathbb{M}_L$. Since h is strictly increasing

$$\frac{h(\varphi(t)) - h(\phi(t))}{\varphi(t) - \phi(t)} = \frac{h(\phi(t)) - h(\varphi(t))}{\phi(t) - \varphi(t)} > 0, \tag{3.11}$$

holds for all $t \in D(\phi, \varphi)$. By (H3), we have

$$1 \geq \sup_{t\in U_L} h'(t) \geq \inf_{s\in U_L} h'(s) \geq 0. \tag{3.12}$$

Define the set $U_t \subset U_L$ by $U_t = [\varphi(t), \phi(t)] \cap U_L$ if $\phi(t) > \varphi(t)$, and $U_t = [\phi(t), \varphi(t)] \cap U_L$ if $\phi(t) < \varphi(t)$, for $t \in D(\phi, \varphi)$. Hence, for a fixed $t_0 \in D(\phi, \varphi)$ we get by (H4) and (3.11) that

$$\sup\{h'(u) : u \in U_{t_0}\} \geq \frac{h(\phi(t_0)) - h(\varphi(t_0))}{\phi(t_0) - \varphi(t_0)} \geq \inf\{h'(u) : u \in U_{t_0}\}.$$

Since $U_t \subset U_L$ for every $t \in D(\phi, \varphi)$, we find

$$\sup_{u \in U_L} h'(u) \geq \sup\{h'(u) : u \in U_{t_0}\} \geq \inf\{h'(u) : u \in U_{t_0}\} \geq \inf_{u \in U_L} h'(u),$$

and therefore,

$$1 \geq \sup_{u \in U_L} h'(u) \geq \frac{h(\varphi(t)) - h(\phi(t))}{\varphi(t) - \phi(t)} \geq \inf_{u \in U_L} h'(u) \geq 0 \tag{3.13}$$

for all $t \in D(\phi, \varphi)$. So, (3.13) yields

$$\begin{aligned} |(H\phi)(t) - (H\varphi)(t)| &= |\phi(t) - h(\phi(t)) - \varphi(t) + h(\varphi(t))| = \\ &= |\phi(t) - \varphi(t)| \Big| 1 - \Big(\frac{h(\phi(t)) - h(\varphi(t))}{\phi(t) - \varphi(t)}\Big) \Big| \leq \\ &\leq |\phi(t) - \varphi(t)| \Big(1 - \inf_{u \in U_L} h'(u)\Big) \end{aligned} \tag{3.14}$$

for all $t \in D(\phi, \varphi)$. Thus, (3.13) and (3.14) imply that H is a large contraction in the supremum norm. To see this choose a fixed $\epsilon \in (0, 1)$ and assume that ϕ and φ are two functions in $\mathbb{M}_L$ satisfying

$$\|\phi - \varphi\| = \sup_{t \in [-L, L]} |\phi(t) - \varphi(t)| \geq \epsilon.$$

If $|\phi(t) - \varphi(t)| \leq \epsilon/2$ for some $t \in D(\phi, \varphi)$, then from (3.14)

$$|(H\phi)(t) - (H\varphi)(t)| \leq |\phi(t) - \varphi(t)| \leq \frac{1}{2} \|\phi - \varphi\|. \tag{3.15}$$

Since h is continuous and strictly increasing, the function $h(u + \frac{\epsilon}{2}) - h(u)$ attains its minimum on the closed and bounded interval $[-L, L]$. Thus, if $\frac{\epsilon}{2} < |\phi(t) - \varphi(t)|$ for some $t \in D(\phi, \varphi)$, then from (3.13) and (H3) we conclude that

$$1 \geq \frac{h(\phi(t)) - h(\varphi(t))}{\phi(t) - \varphi(t)} > \lambda,$$

and therefore,

$$\begin{aligned} |(H\phi)(t) - (H\varphi)(t)| &\leq |\phi(t) - \varphi(t)| \Big\{1 - \frac{h(\phi(t)) - h(\varphi(t))}{\phi(t) - \varphi(t)}\Big\} \leq \\ &\leq (1 - \lambda) \|\phi(t) - \varphi(t)\|, \end{aligned} \tag{3.16}$$

where

$$\lambda := \frac{1}{2L} \min \left\{ h\Big(u + \frac{\epsilon}{2}\Big) - h(u), u \in [-L, L] \right\} > 0.$$

Consequently, it follows from (3.15) and (3.16) that

$$|(H\phi)(t) - (H\varphi)(t)| \leq \delta \|\phi - \varphi\|,$$

where $\delta = \max\left\{\frac{1}{2}, 1 - \lambda\right\} < 1$. The proof is complete. □

The next result gives a relationship between the mappings H and B in the sense of a large contraction.

Lemma 3.5. *If H is a large contraction on $\mathbb{M}_L$, then so is the mapping B.*

Proof. If H is a large contraction on $\mathbb{M}_L$, then for $x, y \in \mathbb{M}_L$, with $x \neq y$, we have $\|Hx - Hy\| \leq \|x - y\|$. Thus, it follows from the equality

$$a(u)e^{-\int_u^t a(r)dr} = \frac{d}{du}\left[e^{-\int_u^t a(r)dr}\right],$$

that

$$|Bx(t) - By(t)| \leq \left(1 - e^{-\int_{t-T}^{t} a(s)ds}\right)^{-1} \times \int_{t-T}^{t} a(u)\big|H(x(u)) - H(y(u))\big| e^{-\int_u^t a(r)dr} du \leq$$

$$\leq \frac{\|x - y\|}{\left(1 - e^{-\int_{t-T}^{t} a(s)ds}\right)^{-1}} \int_{t-T}^{t} a(u)e^{-\int_u^t a(r)dr} du = \|x - y\|.$$

Thus,

$$\|Bx - By\| \quad \leq \quad \|x - y\|.$$

One may also show in a similar way that

$$\|Bx - By\| \quad \leq \quad \delta\|x - y\|$$

holds if we know the existence of a $\delta \in (0, 1)$ and that for all $\epsilon > 0$

$$\big[x, y \in \mathbb{M}_L,\ \|x - y\| \geq \epsilon\big] \ \Rightarrow \|Hx - Hy\| \leq \delta\|x - y\|.$$

The proof is complete. □

By Lemma 2.1, φ is a solution of (1.1) if

$$\varphi = A\varphi + B\varphi,$$

where B and A are given by (3.2) and (3.3) respectively. By Lemma 3.2, $A : \mathbb{M}_L \to \mathbb{M}_L$ is completely continuous. By Lemma 3.3, $A\varphi + B\psi \in \mathbb{M}_L$ whenever $\varphi, \psi \in \mathbb{M}_L$. Moreover, $B : \mathbb{M}_L \to \mathbb{M}_L$ is a large contraction by Lemma 3.5. Thus all the hypotheses of Theorem 2.4 are satisfied. Thus, there exists a fixed point $\varphi \in \mathbb{M}_L$ such that $\varphi = A\varphi + B\varphi$. Hence (1.1) has a $T-$ periodic solution.

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