TOTAL LOGISTIC MANAGEMENT No. 2 ● 2009 ● PP. 27–36

Waldemar KACZMARCZYK*

INVENTORY COST SETTINGS IN SMALL BUCKET LOT-SIZING AND SCHEDULING MODELS

Abstract: This paper addresses models for lot-sizing and scheduling of several products on machines with limited capacity. Small time buckets in such models may arise from subdividing real (macro-)periods into several fictitious micro-periods. The inventory holding costs may then be accounted for either at the end of every micro-period or only at the end of the macro-period. Presented results of computational experiments show that these two settings lead to completely different models and solutions. After reviewing the definition of inventory holding costs, we propose a rule for correctly setting the unit costs which is essential for a correct formulation of the model.

Keywords: production, lots sizing and scheduling.

1. Introduction

This paper addresses models for *lot-sizing and scheduling* of several products on machines with limited capacity. They may be divided into two classes (Drexl, Kimms 1997, Pochet, Wolsey 2006, Jans, Degraeve 2008). The so-called *large bucket* models permit multiple products to be produced in each period, but do not make any assumption about the sequence of orders within a single period. A large time bucket typically represents a time slot of one week and the planning horizon is less than six months.

Subdividing the macro-periods of a large bucket model into several fictitious micro-periods leads to the so-called *small bucket* models. These models assume that within each period only one set-up operation may be executed and therefore at most two products may be produced, the first before and the second after the set-up operation. Hence, periods in these models usually correspond to small time slots such as hours or shifts.

An often disregarded difference between large and small bucket models is the precision in accounting of inventory holding costs. While a large bucket model may

* AGH University of Science and Technology, Krakow, Poland

account for the inventory cost only at the end of macro-periods, a small bucket model may do this at the end of every micro-period. It is an imposing idea to account for inventory holding cost equally in all micro-periods. One may believe this is simply more detailed, i.e. more accurate, cost accounting. But the analysis presented leads to a different conclusion.

To understand the consequences of various settings of inventory holding costs, the results of two models are compared. *The Capacitated Lot-sizing Problem with Linked lots* (CLSPL) is an extension of the classic CLSP, the best known large bucket model. The CLSP in every period schedules a set-up operation for every processed product. The CLSPL however allows a single lot to overlap adjacent periods, i.e. to start a lot in one period and to finish it in the next period (Haase 1998, Sox, Gao 1999, Suerie, Stadtler 2003). Such lot linking allows significant savings. Haase (1998) reports a reduction of the total costs of up to 16% and Suerie (2005) even 20%.

The Proportional Lot-sizing and Scheduling Problem (PLSP) is the most flexible among the small bucket models, as it allows lots of any size to overlap several periods and allows to process two products within a single period, one before and another after the set-up operation (Drexl, Haase 1995). For a review of lot-sizing and scheduling models, see Drexl and Kimms (1997), Pochet and Wolsey (2006), Jans and Degraeve (2008) and Quadt and Kuhn (2008).

A first attempt to approximate a large bucket model with a small bucket model was made by Fleischmann (1990). He compared the classic CLSP (without linked lots) to the Discrete Lot-sizing and Scheduling Problem (DLSP), which only allows lots to be multiples of micro-period capacity. Therefore, to achieve high accuracy, it was necessary to use very small periods. Suerie (2005) compared the CLSPL with the PLSP, but used periods of the same length in both models. For instance, with low set-up costs, i.e. many small lots and frequent set-up operations in optimal solutions, it is very likely that a small bucket model will not be able to schedule as many set-ups as necessary. For this reason Suerie (2005) reported that significantly higher total costs were obtained with the small bucket model.

In this paper, an approximation will be achieved using several sub-periods of large buckets. As has already been indicated in Kaczmarczyk (2010), splitting macro-periods allows a good approximation of the CLSPL with the PLSP. However this is possible only for a special setting of the inventory holding cost parameters proposed by Fleischman (1990), who assigned the whole macro-period holding cost to the last sub-period (see also Fleischmann 1994, Brüggemann, Jahnke 2000). Most authors use another setting where the unit cost is identical in all sub-periods (Drexl, Haase 1995, Drexl, Kimms 1997).

An outline of this paper is as follows. At first, in Section 2, two models used in com-putational experiments are characterized. Section 3 presents different ways of accounting for the inventory holding costs in small bucket models. Section 4 describes the results of computational experiments. Section 5 presents a rule for the correct setting of unit inven-tory carrying costs. Finally, Section 6 gives a summary.

2. Models

In Table 1, the parameters and variables for both models discussed in this paper are presented. All variables with time index 0 are equal to 0, except I_{j0} and y_{j0} which describe the initial state of the system. Variables y_{jt} , z_{jt} and w_{jt} are binary.

Table 1. *Parameters and variables*

N	–	set of products,
T	–	set of periods,
SC_j	–	set-up cost of product j,
h_{jt}	–	unit holding cost of product j,
d_{jt}	–	demand of product j in period t,
C_t	–	length of period t,
p_j	–	processing time of product j,
$ST_j < C_t$	–	set-up time of product j,
I_{jt}	–	inventory of product j in period t, I_{j0} – initial inventory,
x_{jt}	–	production volume of product j in period t,
y_{jt}	–	1, if in period t machine is *set up* to process product j, 0 otherwise, y_{j0} – initial state,
z_{jt}	–	1, if in period t machine *starts up* to process product j, 0 otherwise,
w_{jt}	–	1, if lots of product j in periods $t-1$ and t are *linked*, i.e. are executed consecutively after a single set-up operation, 0 otherwise.

The model of the Capacitated Lot-Sizing Problem with Linked lots in adjacent periods (CLSPL) proposed by Sox and Gao (1999) is described below.

$$\text{Minimise:} \quad \sum_{j\in N}\sum_{t\in T}\left(SC_j z_{jt} + h_{jt} I_{jt}\right) \tag{1a}$$

$$\text{subject to:} \quad I_{j,t-1} + x_{jt} - d_{jt} = I_{jt}, \qquad t \in T, j \in N \tag{1b}$$

$$\sum_{j\in N}\left(p_j x_{jt} + ST_j z_{jt}\right) \leqslant C_t, \qquad t \in T \tag{1c}$$

$$x_{jt} \leqslant A_{jt}\left(z_{jt} + w_{jt}\right), \quad t \in T, j \in N \tag{1d}$$

$$\sum_{j\in N} w_{jt} = 1, \qquad t \in T \tag{1e}$$

$$w_{jt} \leqslant z_{j,t-1} + w_{j,t-1}, \quad t \in T, j \in N \tag{1f}$$

$$w_{jt} + w_{j,t-1} + z_{k,t-1} \leqslant 2, \qquad t \in T, j \in N : j \neq k \tag{1g}$$

$$w_{j0} \leqslant y_{j0}, \qquad t \in T, j \in N : j \neq k \tag{1h}$$

$$z_{jt}, w_{jt} \in \{0, 1\}, \quad t \in T, j \in N \tag{1i}$$

$$x_{jt}, I_{jt} \geqslant 0, \qquad t \in T, j \in N \tag{1j}$$

where $A_{jt} = min\left(C_t/p_j, \sum_{s=t}^{T} d_{js}\right)$.

The objective function (1a) is defined to minimize the sum of set-up and inventory holding costs. Constraints (1b) describe the balance of inventory, production and demand. Constraints (1c) ensure that the total workload does not exceed capacity. Constraints (1d) allow production to take positive values only if the machine is set up for a given item.

Constraints (1e-h) allow the start of a lot in one period and the finish in the next period, known as lot liking (Haase 1998) or set-up carryover (Sox, Gao 1999). Constraints (1e) ensure that only one product may be scheduled at the end of a period. Constraints (1f) ensure that the set-up of product which was not produced in the previous period is not carried over to the current period. In this case, when the set-up of the same product is carried over from period $t-2$ to periods $t-1$ and t, constraints (1g) prevent a start-up of any other product in period $t-1$. Constraints (1h) set the initial state of the machine.

The classic model of the Proportional Lot-sizing and Scheduling Problem (PLSP) proposed by Drexl and Haase (1995) is presented below.

Minimise: $$\sum_{j \in N} \sum_{t \in T} \left(SC_j z_{jt} + h_{jt} I_{jt}\right) \tag{2a}$$

subject to: $$I_{j,t-1} + x_{jt} - d_{jt} = I_{jt}, \qquad t \in T, j \in N \tag{2b}$$

$$\sum_{j \in N} \left(p_j x_{jt} + ST_j z_{jt}\right) \leqslant C_t, \qquad t \in T \tag{2c}$$

$$p_j x_{jt} \leqslant C_t \left(y_{j,t-1} + y_{jt}\right), \quad t \in T, j \in N \tag{2d}$$

$$\sum_{j \in N} y_{jt} \leqslant 1, \qquad t \in T \tag{2e}$$

$$y_{jt} - y_{j,t-1} \leqslant z_{j,t}, \qquad t \in T, j \in N \tag{2f}$$

$$y_{jt} \in \{0, 1\}, \qquad t \in T, j \in N \tag{2g}$$

$$x_{jt}, z_{jt}, I_{jt} \geqslant 0, \qquad t \in T, j \in N \tag{2h}$$

Objective function (2a) and first two constraints (2b-c) are the same as in the CLSPL. Constraints (2d) allow production only if the machine is set up for a given product in the current or previous period. Constraints (2e) ensure that the machine in a single period is set-up for at most one product. Constraints (2f) preserve accurate values for the start-up variables. The start-up variables do not have to be defined as binary because their values are limited only by this binary expression.

Quadt and Kuhn (2009) proposed an extension of the CLSPL for identical parallel machines and Kaczmarczyk (2010) proposed similar extension for the PLSP. Although in the next sections, the results of these models will be compared, further details of them are omitted as they are not important for the issues discussed.

Finally, one difference between the CLSPL and PLSP needs to be emphasized again. While periods in the CLSPL are usually longer than a day, e.g. weeks or months, in the PLSP periods are usually shorter, e.g. hours or shifts. Therefore, if someone would like to apply both the CLSP and PLSP to solve the same problems he would have to define much more periods in the PLSP.

3. Modeling inventory holding costs

To solve large bucket instances with a small bucket model, it is necessary to prepare the demand and unit holding cost parameters in a special way. Firstly, the large buckets (macro-periods) have to be split into several sub-periods (micro-periods). Let us assume that the number of sub-periods is equal to 5. Next, the demand of every macro-period has to be assigned to its last micro-period so that only micro-periods which are multiples of 5 have non-zero demand.

Similarly, the unit inventory holding costs h_{jt} have to be set equal to zero in all micro-periods with the exception of periods that are multiples of 5. Such parameter setting will be further referred to as macro-period cost setting (*macro-setting*). Solutions obtained with the PLSP in this way will be called *large bucket solutions*, as they approximate the solution of a large bucket model.

The unit inventory holding cost h_{jt} could also be set non-zero for all micro-periods but five times smaller than for a single macro-period. Such micro-period cost setting (*micro-setting*) used in typical small bucket models would make it impossible to compare the results of the two models. Solutions obtained in this way will be called *small bucket solutions*. An example of micro- and macro-period cost settings is presented in Table 2.

Table 2. *Example of inventory holding cost settings*

Macro-periods			1					2		
Micro-periods	1	2	3	4	5	6	7	8	9	10
Macro-setting	0	0	0	0	5	0	0	0	0	5
Micro-setting	1	1	1	1	1	1	1	1	1	1

The macro- and micro-settings lead to different models but this difference has until now been disregarded. In the next section, the large and small bucket solutions of several datasets are compared to give a numerical illustration of this fact and to verify our intuitive understanding of its consequences.

4. Computational results

Firstly one has to describe in more detail the datasets (instances) prepared by Quadt and Kuhn (2009). Unit process times were generated using a uniform distribution between 5 and 15, i.e. $U(5, 15)$, initial inventory volumes were set to zero, and the initial set-up state was drawn at random for each machine. Set up times were drawn from a uniform distribution between 5% and 15% of the average capacity usage per period of an average product. Set-up costs were drawn from $U(3000, 5000)$.

Five factors of the problem data were varied, i.e. there were $2 \times 2 \times 2 \times 2 \times 3 = 48$ instances:

1) *Cost structure:* Low inventory costs $U(1, 20)$, high $U(5, 100)$.
2) *Capacity utilization:* Low 70%, high 85%.
3) *Demand range:* In a low variance case, the demand values were drawn from $U(90, 110)$, in high variance case from $U(50, 150)$.
4) *Periods with positive demand:* In one case, a positive demand was generated for every product and period. In another case, on average every second period had a positive demand, however, a mean of 100 units per period was kept.
5) *Demand pattern:* Three scenarios were considered: equally distributed demands among products and periods, an unequal demand distribution among products (high and low volume products), and a positive demand trend.

Quadt and Kuhn (2009) prepared and solved two types of instances: small and large ones. In this paper, only the small instances with 5 products, 4 periods and 4 machines have been used. All of them could be solved to optimality for both models. As the number of products per machine was 1.25, it was assumed that splitting macro-periods into 5 micro-periods would allow comparable results to be obtained with both models.

All instances have already been solved by Kaczmarczyk (2010) using the CLSPL and the PLSP with macro-period cost setting, i.e. the large bucket solutions were obtained. In general, one could expect that the large bucket model would preserve better solutions, because in the small bucket model lot sizes and utilization of machines are constrained in a specific way by the length of the micro-periods.

It turned out that in 29 out of 48 solved cases, the optimal value of the objective function was exactly the same for both models. On average, the PLSP got only 0.2% higher values than the CLSPL and in the worst case 2.3%. Despite all the differences, both models deliver almost identical results for macro-period cost setting.

Solutions obtained for the PLSP by macro-setting, i.e. *large bucket solutions*, are optimal for the macro-setting. Solutions obtained by micro setting, i.e. *small bucket*

solutions, are optimal for the micro-setting. But they are completely different, because they are obtained for different models, and applying them in the wrong cost setting leads to significant errors. To estimate these errors, all instances have been solved once again but now by micro-setting, i.e. this way we obtained small bucket solutions. Results are presented in Table 3. *The approximation error*, i.e. the relative difference between optimal objective values of the large and small bucket model in macro-setting is negligible for all cases.

Table 3. *Comparison of the big bucket and the small bucket solutions*

Parameter	Value	Approximation error	Big bucket error	Small bucket error
Inventory cost	Small: [1–20]	0.20%	13%	16%
	High: [5–100]	0.20%	19%	49%
Utilization	Small: 70%	0.10%	24%	47%
	High: 85%	0.30%	12%	29%
Demand range	Narrow: [1–20]	0.10%	23%	43%
	Wide: [5–100]	0.30%	13%	32%
Periods with positive demand	All: 100%	0.40%	22%	55%
	Random: 50%	0.10%	13%	24%
Demand pattern	Equal distribution	0.20%	21%	51%
	High & low volume products	0.20%	21%	53%
	Positive trend	0.20%	11%	16%
	Average	0.20%	17%	37%

The additional total cost of the large bucket solution in micro-setting related to the small bucket solution is on average equal to 17%. Let us call this *the large bucket error*. We make this error when periods relevant for the planning are aggregated due to the complexity of the model. Such a large error is a good justification for additional effort which may be necessary in order to solve models with a more detailed planning calendar, e.g. small bucket models.

It is not surprising that the large bucket error is higher for high unit inventory holding costs. It is also smaller for high utilization which may be explained by the smaller solution space, which makes it harder to determine different solutions. Wide demand range and randomly chosen periods with positive demand lead to higher demand variance and smaller large bucket error. A possible explanation could be the fact that higher variance may create a specific demand pattern which forces partly similar solutions in both models.

The small bucket error, i.e. the relative error of the small bucket solution in macro-setting is on average 37%. The meaning of this error is explained in the next section. Table 3 shows that the small bucket error depends on all the parameters in a similar

way to the large bucket error, except that the scale of this error is almost two times higher. All discrepancies between the macro- and micro-period cost calculations can be easily explained by the fact that the large bucket model optimizes only the costs at the end of the macro-periods and ignores expensive inventories inside them. The small bucket model in-stead uses a detailed time-scale and is not able to hide all these costs.

These results raise important questions: Which of these models (cost settings) is more accurate? Or maybe: Under what circumstances should each model be applied? In the next sections we give an answer to these questions.

5. How to choose inventory holding cost setting

Let us review the basic assumptions of discrete time lot-sizing models. According to Zipkin (2000, p. 80): "The sequence of events at each time point t is as follows: We observe the inventory I_t and decide to order size x_t. Then, sometime during period t, the order x_t arrives and the demand d_t occurs. We don't care precisely when these events happen, provided they are complete by the end of the period, in time to determine I_{t+1} at time point $t+1$."

According to Smith (1989, p. 256): "The carrying cost is based on the inventory available at the end of each planning period. This will be equal to the actual inventory throughout the period if demand occurs instantaneously at the beginning of the period just after receipt of the replenishment order if there is one." Even if demand does not satisfy this assumption "the resulting additional inventory is unaffected by choice of an ordering policy and so can be disregarded."

It seems that only the time points at which receipt of an order is possible are important. These points are defined by the smallest time units used to describe the replenishment and delivery due dates. Let us call them *the receipt periods*. In a single level model, i.e. without dependent demand, the smallest practical time units are days. Less often, for very long delivery lead times, this could also be weeks.

There now follows the most important conclusion in this article. How lots are scheduled within such periods is completely unimportant for real inventory cost accounting, as it has no impact on replenishment or distribution decisions, or the real cost that arises. Therefore the correct setting should assign positive values of inventory cost to micro-periods which end at the same time points as receipt periods.

How does this match the macro- and micro-periods? If the macro-periods are equal to the receipt periods, then the macro-setting is appropriate, as it accounts only for inventory at the end of all macro-periods. Using the micro-setting would lead to fictitious optimization of inventory costs within the macro-period at the cost of the real costs. This is *the small bucket error* estimated in the previous section, i.e. the relative error of the small bucket solution in macro-setting.

This error is made for example when inventory carrying costs are accounted for after every shift, although both replenishment and delivery are planned on a daily calendar, and the schedule within a day has no impact on any real costs. If the micro-periods are equal to the receipt periods, then the micro-setting is appropriate, as it

allows to minimize inventory costs within every macro-period. For example, a small bucket model with micro-setting will be a natural choice if buckets are equal to days, demand is defined on a daily calendar and the assumption of at most one set-up operation within a single period is fully justified. Applying large bucket models in such circumstances, by aggregation of days into larger time buckets, might however be necessary to reduce the complexity of the problem. The aggregation however causes an error called *the large bucket error*.

If the receipt periods are smaller than the macro-periods, but longer than the micro-periods, it will be necessary to apply a mixed setting: only costs of the last micro-periods, within a receipt period, should have positive value. This is identical to macro-setting.

6. Summary

In this paper, correct modelling of the inventory holding costs in lot-sizing and scheduling models is considered. Two models are discussed, a large bucket model, the CLSPL, and a small bucket model, the PLSP. A comparison of the large bucket and small bucket models was made for two settings of the unit inventory holding cost parameters: first, when inventory costs are accounted for only at the end of the macro-periods, and second, when inven-tory costs are accounted for in every micro-period. This comparison shows that these solutions are not equivalent which raises the question, which of them is the more accurate one.

A rule proposed for the setting of inventory carrying cost parameters is based on time points when the due dates of orders may be set. Correct parameter setting allows inventory cost accounting only at these points, because only these costs may have an impact on replenishment or distribution decisions, i.e. only these costs are real.

A future research topic may be to develop accurate parameter settings in a multi-level environment. Should the inventory of final products be accounted for on the same scale as the inventory of subassemblies? Another question that remains open is: how to choose the number of sub-periods to balance problem complexity and approximation error?

Finally, if unit inventory holding costs are equal to zero in all but the last of the micro-periods within a single macro-period, making two start-ups for the same product within that macro-period does not make any sense. This property may be used to modify models, e.g. using Special Ordered Sets, or to develop specialized algorithms.

References

Brüggemann W., Jahnke H. 2000. *The discrete lot-sizing and scheduling problem: Complexity and modification for batch availability*. European Journal of Operational Research, 124, pp. 511–528.

Drexl A., Haase K. 1995. *Proportional lotsizing and scheduling*. International Journal of Production Economics, 40, pp. 73–87.

Drexl A., Kimms A. 1997. *Lot sizing and scheduling – survey and extensions*. European Journal of Operational Research, 99 (2), pp. 221–235.

Fleischmann B. 1990. *The discrete lot-sizing and scheduling problem*. European Journal of Operational Research, 44, pp. 337–348.

Fleischmann B. 1994. *The discrete lot-sizing and scheduling problem with sequence-dependent setup costs*. European Journal of Operational Research, 75, pp. 395–404.

Haase K. 1998. *Capacitated lot-sizing with linked production quantities of adjacent periods*. [in:] Drexl A., Kimms A. (Eds), Beyond manufacturing resource planning (MRP II), Springer, Berlin, pp. 127–146.

Jans R., Degraeve Z. 2008. *Modeling industrial lot sizing problems: a review*. International Journal of Production Research, 46 (6), pp. 1619–1643.

Kaczmarczyk W. 2009. *Proportional Lot-sizing and Scheduling Problem with Identical Parallel Machines*. Submitted to International Journal of Production Research (unpublished).

Pochet Y., Wolsey L. A. 2006. *Production planning by mixed integer programming*. Series in Operations Research and Financial Engineering, Springer, New York.

Quadt D., Kuhn H. 2009. *Capacitated Lot-Sizing and Scheduling with Parallel Machines*. Back-Orders and Setup Carry-Over, Naval Research Logistics, 56, pp. 366–384.

Quadt D., Kuhn H. 2008. *Capacitated lot-sizing with extensions: a review*. 4OR, 6, pp. 61–83

Smith S.B. 1989. *Computer-Based Production and Inventory Control*. Prentice Hall, Englewood Cliffs.

Sox C.R., Gao Y. 1999. *The capacitated lot sizing problem with setup carry-over*. IIE Transactions, 31, pp. 173–181.

Suerie Ch. 2005. *Time Continuity in Discrete Time Models New Approaches for Production Planning in Process Industries*. Springer, Berlin, Heidelberg.

Zipkin P.H. 2000. *Foundations of Inventory Management*. McGraw-Hill, Boston.